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Large cardinals and structural reflection

Romanos E. Aslanis-Petrou

Supervisor: **Konstantinos Tsaprounis**, Assistant Professor, Department of Mathematics, University of the Aegean, Samos, Greece.

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Ρωμανός Ε. Ασλάνης-Πέτρου

Επιβλέπων: Κωνσταντίνος Τσαπρούνης, Επίκουρος Καθηγητής, Τμήμα Μαθηματικών, Πανεπιστήμιο Αιγαίου, Καρλόβασι, Σάμος.

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Romanos E. Aslanis-Petrou
S.N.: 7115142200003

SUPERVISOR: **Konstantinos Tsaprounis**, Assistant Professor, Department of Mathematics, University of the Aegean, Samos, Greece.

EXAMINATION COMMITTEE:
Gregoriades Vassilios, Assistant Professor, School of Applied Mathematical and Physical Sciences, Department of Mathematics, National Technical University of Athens, Greece.
Souldatos Ioannis, Assistant Professor, Department of Mathematics, Aristotle University of Thessaloniki, Greece.
Tsaprounis Konstantinos, Assistant Professor, Department of Mathematics, University of the Aegean, Samos, Greece.

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ΕΞΕΤΑΣΤΙΚΗ ΕΠΙΤΡΟΠΗ:
Γρηγοριάδης Βασίλειος, Επίκουρος Καθηγητής, Σχολή Εφαρμοσμένων Μαθηματικών και Φυσικών Επιστημών, Εθνικό Μετσόβιο Πολυτεχνείο.
Σουλδάτος Ιωάννης, Επίκουρος Καθηγητής, Τμήμα Μαθηματικών, Αριστοτέλειο Πανεπιστήμιο Θεσσαλονίκης.
Τσαπρούνης Κωνσταντίνος, Επίκουρος Καθηγητής, Τμήμα Μαθηματικών, Πανεπιστήμιο Αιγαίου, Καρλόβασι, Σάμος.

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ABSTRACT

The axiomatic system of first-order ZFC set theory constitutes one of the most prominent bases for mathematics; at least for classical ones. However, after the “discovery” of Cohen’s forcing technique, a plethora of mathematical problems have been proved to be independent of these axioms, thereby suggesting that the search for new axioms for mathematics is an issue of paramount importance. One of the most prominent categories of such axioms are the so-called large cardinal axioms which, up to this day, are playing a pivotal role in “eliminating” some of these independence phenomena. Moreover, not only there have been unveiled deep connections between such axioms and various areas of mathematics, but also it has been observed that these postulates form a hierarchy which can be used to “measure” the consistency strength of several other axioms that have been proposed. Now, in this thesis we first make a brief introduction to the theory of large cardinals, outlining that way the basic concepts and tools we will be using, as well as commenting upon some intriguing related issues. Subsequently, we follow the work of Bagaria in [2] and we focus our attention on the notions of (some) $C^{(n)}$-cardinals; especially on that of $C^{(n)}$-extendibles. Moving to the final part and the core of our study, we investigate the area in between supercompact cardinals and Vopěnka’s Principle, where a level-by-level correspondence between the hierarchy of $C^{(n)}$-extendible cardinals and strata of Vopěnka’s Principle is uncovered, as presented in Sn 4 of [2].

SUBJECT AREA: Set Theory

KEYWORDS: Large cardinals, $C^{(n)}$-cardinals, $C^{(n)}$-extendible cardinals, Vopěnka’s Principle

ΠΕΡΙΛΗΨΗ

Το αξιωματικό σύστημα της πρωτοβάθμιας ZFC συνολοθεωρίας αποτελεί μία από τις κυρίαρχες βάσεις για τα μαθηματικά, τουλάχιστον για αυτά που βασίζονται στην κλασική λογική. Ωστόσο, μετά την “ανακάλυψη” της τεχνικής του forcing από τον Cohen, πληθώρα μαθηματικών προβλημάτων αποδείχθηκαν ανεξάρτητα από τα αξιώματα της ZFC, υποδηλώνοντας ότι η αναζήτηση νέων αξιωμάτων είναι ένα ζήτημα πρωταρχικής σημασίας. Μία από τις κυρίαρχες κατηγορίες νέων αξιωμάτων είναι τα λεγόμενα αξιώματα μεγάλων πληθαρίθμων, τα οποία μέχρι και σήμερα παίζουν καθοριστικό ρόλο στην “εξάλειψη” ορισμένων φαινόμενων ανεξαρτησίας. Επιπλέον, όχι μόνο έχουν αποκαλυφθεί βαθιές συνδέσεις μεταξύ αυτών των αξιωμάτων και διαφόρων πεδίων των μαθηματικών, αλλά έχει παρατηρηθεί ότι τα αξιώματα μεγάλων πληθαρίθμων σχηματίζουν μία ιεραρχία που μπορεί να χρησιμοποιηθεί για να “μετρηθεί” η ισχύς συνέπειας των διαφόρων άλλων αξιωμάτων που έχουν προταθεί. Σε αυτή τη διπλωματική, αρχικά παρουσιάζουμε μια σύντομη εισαγωγή στη θεωρία των μεγάλων πληθαρίθμων, περιγράφοντας τις βασικές έννοιες και τα βασικά εργαλεία που θα χρησιμοποιήσουμε, καθώς και σχολιάζουμε ορισμένα ενδιαφέροντα συναφή ζητήματα. Στη συνέχεια, ακολουθώντας την έρευνα του Bagaria στο [2], επικεντρωνόμαστε σε ορισμένες έννοιες $C^{(n)}$-πληθαρίθμων και, ειδικά, σε αυτή των $C^{(n)}$-extendible. Προχωρώντας στο τελικό και κύριο μέρος της μελέτης μας, εξερευνούμε την περιοχή μεταξύ των supercompact πληθαρίθμων και της Αρχής του Vopěnka, όπου μια επίπεδο-προς-επίπεδο αντιστοιχία μεταξύ της ιεραρχίας των $C^{(n)}$-extendible πληθαρίθμων και επιπέδων της Αρχής του Vopěnka αποκαλύπτεται, όπως παρουσιάζεται στην Ενότητα 4 του [2].

ΘΕΜΑΤΙΚΗ ΠΕΡΙΟΧΗ: Θεωρία Συνόλων

ΛΕΞΕΙΣ ΚΛΕΙΔΙΑ: Αξιώματα μεγάλων πληθαρίθμων, $C^{(n)}$-πληθάριθμοι, $C^{(n)}$-extendible πληθάριθμοι, Αρχή του Vopěnka

CONTENTS

1. INTRODUCTION

During the 1870's, having as a starting point the study of various subsets of real numbers, Cantor began an investigation of the concept of mathematical infinity. One of his most famous results was that there can be no bijection between the set of real numbers and that of the natural numbers and, more generally, that there can be no bijection between any set X and the set of all subsets of X, i.e., the powerset of X. Now, in the case of finite sets, the existense of a bijection between two finite sets X and Y can be seen as an adequate formalization of the intuitive concept of comparing their sizes. To be exact, if such a bijection exists, then it is natural to say that the sets X and Y have the same number of elements. Cantor's conceptual innovation was the generalization of this idea to the case of infinite sets, from which ultimately follows that there is an infinitude of different sizes of infinities. Hence, the voyage into the transfinite began and set theory, the study of infinite sets, was born.

However, while most mathematicians and logicians were intrigued by Cantor's results, some others rejected his methods and heavily criticized them. Furthermore, around the beginning of the 20th century, Russell was digging out paradoxes both in Cantor's (naive) set theory and Frege's attempt in reducing mathematics to logic. In the following years, it became clear that a common factor of the above issues was the lack of a proper formalization of mathematics and, to this direction, a great number of mathematicians and logicians worked hard. Skipping a lot of historical (as well as mathematical) details, they were eventually led to formalize mathematics in first-order predicate logic and, in particular, in the language of set theory.

Hence, after further elaboration over the years, the axiomatic system of first-order ZFC, i.e., Zermelo-Fraenkel set theory with the Axiom of Choice, emerged and to this day, it constitutes one of the most prominent axiomatic systems for (classical) mathematics, in the sense that almost every mathematical theory can be interpreted inside it. However, dangers were still lurking. In 1931, Gödel's famous incompleteness theorems showed that every "appropriate"[1] and *consistent* theory T, i.e. free of contradictions, is *incomplete*, i.e. there are mathematical statements that T can not prove nor refute (statements of this sort are called *independent* of the axioms of T). In addition, he proved that, if T is indeed a consistent theory, then the consistency of T is such an independent statement. In other words, there is no hope to provably guarantee, within T, that no contradiction will emerge from the theory T. So, for ZFC in particular, our best hope is that it will withstand the trial of time and, at the same time, that it will be adequate for encompassing future mathematics.

Gödel's incompleteness theorems shocked the mathematical community of that time but, shortly after, mathematicians started exploring deeper such meta-mathematical issues and came to the understanding that not everything was lost. With Gödel's gift (or curse) which gave a clearer general picture of mathematics, the work on the foundations continued and various interesting results were brought to the surface. Furthermore, the majority of the "working" mathematicians were ignoring Gödel's "threat" of incompleteness and,

[1]By appropriate we mean recursively axiomatizable and capable of expressing at least basic arithmetic.

disregarding such issues, continued their research. However, in the following years, a new "discovery" would once again change "the rules of the game".

In 1963, Paul Cohen introduced the powerful technique of *forcing* and established the relative consistency of the negation of the *Continuum Hypothesis* (CH) with ZFC. At this point, let us mention that CH is a famous conjecture stated by Cantor himself in 1878 that, informally, states that there is no set whose cardinality is strictly between that of the integers and that of the real numbers. Now, already in 1938, Gödel had shown that CH holds in his constructible universe L (which implies that CH is relatively consistent with ZFC) and thus, together with Cohen's result, it follows that CH is a statement (of mathematics) that is independent of ZFC, i.e., the truth value of CH can not be established on the basis of ZFC. Moreover, in the following years, using Cohen's method a plethora of interesting problems from almost every area of mathematics turned out to be independent of ZFC, hinting that way that ZFC, if consistent, is not powerful enough. Hence, in order to establish the truth or the falsity of these statements, new axioms strengthening ZFC should be found.

Over the course of time, there have been proposed dozens, if not hundreds of such new axioms; ranging from combinatorial principles (e.g. *Diamond* ($\diamondsuit$), *Square* ($\square$)) to axioms of a game-theoretic origin (e.g. *Axiom of Determinacy* (AD)), to forcing axioms (e.g. *Proper Forcing Axiom* (PFA), *Martin's Maximum* (MM)) and *large cardinal axioms* (which, some of them, we will shortly present). The list of these axioms starts with axioms with mild consequences and ends up with very powerful postulates that have tremendous implications in set theory, as well as mathematics in general.

Now, large cardinals are postulates that assert the existence of cardinals that, in a way, are strong forms of infinity. An initial motivation for their definition (at least for some of them), was the generalization of properties of ω (the least infinite set) to higher cardinalities. For example, ω is a regular and strong limit cardinal, properties which, when required by an uncountable cardinal lead to the concept of an *inaccessible* cardinal. Over the years, an empirical phenomenom that has been observed is that large cardinals form a hierarchy[2] that is linearly ordered in terms of consistency strength, providing us with a scale of "measuring" the strength of various other axioms. For example, (the consistency of) the existence of a *supercompact* cardinal implies the consistency of *Martin's Maximum*. For another instance, the existence of infinitely many *Woodin* cardinals implies the axiom of *Projective Determinacy*.

It should be noted that, the starting point of the large cardinal axioms, the inspiration behind them, can be traced back to Gödel and even Cantor himself. On the one hand, they are driven by the concept of *maximality*; an aspect that suggests that "the" set-theoretic universe should be as "tall" and as "wide" as possible, that way including as many sets and as much information about them as it possibly can. On the other hand, we have the general concept of set-theoretic *reflection*. An instance of this latter concept is incarnated

[2]At the end of the introduction, we have created a figure of that hierarchy that contains all the large cardinal notions we will be presenting in this thesis. However, keep in mind that there are many more axioms than those presented here.

by the well-known *Principle of Reflection* of Lévy and Montague which roughly states that, if a first-order formula holds in the universe V, then it also holds in some initial segment of it. This principle has been a crucial ingredient, or at least the main motivation, in various set-theoretic topics and, in particular, ours.

Travelling a bit in time, we now come to the late 2000's and early 2010's, at the period during which Bagaria introduced the concept of $C^{(n)}$-cardinals; a notion that lies at the intersection of the aforementioned concepts of large cardinals and reflection. More precisely, as we will later see, $C^{(n)}$-cardinals are stronger forms of some of the usual large cardinal notions that, in addition, possess a particular aspect of reflection. The initial research regarding the notions of $C^{(n)}$-cardinals that was done back then turned out to be fruitful and, in fact, it even started a new program; the program of *structural reflection* which aims at justifying large cardinal axioms in terms of some form of reflection principles.

Now, the content of this thesis is about that initial research. After a brief introduction to the theory of large cardinals, we immediately turn to the concept of $C^{(n)}$-cardinals and explore (some of) their properties. More precisely: in Chapter 1, after fixing the language we will be writing with, we present the general tools we will be using and the most basic large cardinal notions; that of an inaccessible and that of a measurable cardinal. In Chapter 2, we proceed to the stronger large cardinal notions of superstrong, supercompact and extendible cardinals. In Chapter 3, we introduce the ordinal proper classes $C^{(n)}$ and the notions of $C^{(n)}$-superstrong and $C^{(n)}$-extendible cardinals. Moreover, we present the notion of joint supercompactness and superstrongness, which, up to a point, helps us study the connection between $C^{(n)}$-extendible and $C^{(n)}$-supercompact cardinals. Finally, in Chapter 4, which is the core of our study, we explore the area between supercompact cardinals and Vopěnka's principle, where the hierarchy of $C^{(n)}$-extendible cardinals live, providing that way a characterization of $C^{(n)}$-extendibles in terms of the concept of structural reflection.

Let us now mention two things. Firstly, regarding the content of this thesis, apart from Chapters 1 and 2 which contain standard set-theoretic material, the rest of the text presents the work of Bagaria in [2] and a small part of the work of Tsaprounis in [15]. Secondly, knowledge of basic set theory (and some not so basic) is assumed and thus, in case one wants to make a revision, we encourage him to consult the classics [10] and [11]. Moreover, [12] constitutes an outstanding introduction to issues related to absoluteness results, as well as meta-mathematical inquietudes. Finally, for extender-related issues, the reader may consult [11] or [17], but keep in mind that our notation is based on the latter.

With that in mind, if someone feels confident for his (set-theoretic) knowledge, he can easily skip the first two Chapters and immediately jump to Chapter 3. However, he should bare in mind that in Chapters 1 and 2 there are plenty of comments that he might find useful.

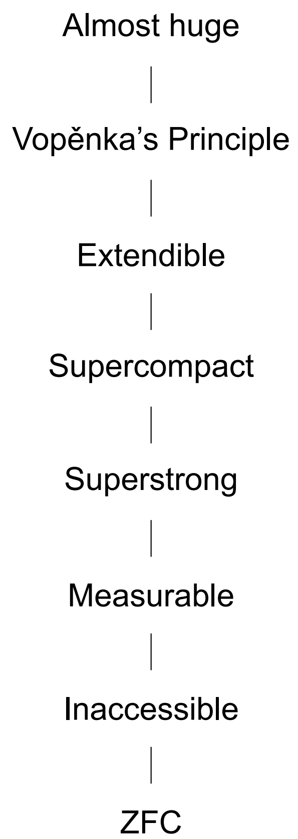


Figure 1.1: The large cardinals notions that show up in this thesis. Their position in the diagram is based on their consistency strength.

2. PRELIMINARIES

In this chapter we give a brief introduction to the theory of large cardinals. We will be presenting the fundamental concepts, grasping this way the opportunity to comment upon various intriguing issues that emerge.

To begin with, let us first fix the (meta-)language, as well as recall some basic tools we will be using.

2.1 Fixing the language

Most of the notation we will be using is quite standard but, in any case, to avoid unpleasant misunderstandings, let us outline some of it.

Unless otherwise stated, we will be working in first-order ZFC. Moreover, it should be noted that some of the theorems that will be presented are in fact schemata (even though they are stated using the word "theorem"). With that being said, a good indication of when we are dealing with a schema of theorems is when, e.g., we write "for $n \geq 0$" or "for $n \geq 1$", where by that we mean a meta-theoretical natural number n. On the other hand, if we write "$n \in \omega$", then n is a (formal set-theoretic) finite ordinal. Now, if T is a formal theory, $Con(T)$ stands for a meta-theoretic assertion of the consistency of T, e.g., T does not prove $\phi \wedge \neg\phi$, for any formula ϕ. Recalling the Lévy hierarchy of formulas, for $n \geq 0$, we say that a formula/class/property is Σ_n (Π_n) if it is Σ_n^{ZFC}-definable (Π_n^{ZFC}-definable) *without* parameters and that it is $\boldsymbol{\Sigma_n}$ ($\boldsymbol{\Pi_n}$) if it is Σ_n^{ZFC}-definable (Π_n^{ZFC}-definable) *with* parameters.

The notation $j : M \prec N$ stands for an elementary embedding j from M into N and if a natural number appears as a subscript, e.g. $j : M \prec_n N$, then it stands for a Σ_n-elementary embedding. Moreover, $M \prec N$ says that M is an elementary substructure of N and, for $n \geq 0$, $M \prec_n N$ that M is a Σ_n-elementary substructure of N. At this point, let us mention that the concept of an elementary embedding is *not* formalizable in first-order ZFC. However, a well-known fact is that if M and N are inner models (of ZFC) such that $j : M \prec_1 N$, then, for every $n \geq 1$, we have that $j : M \prec_n N$. Hence, we can formalize adequately the informal concept of an elementary embedding but, at the same time, "pay the price" of producing schemata of theorems rather than just theorems. Let us mention that we will not be focusing on such matters since the purpose of this thesis is not that but, nevertheless, we find it useful for the reader to have in mind the aforementioned observation. One last thing, the critical point of an elementary embedding j, i.e., the least ordinal moved by j, will be denoted by $cp(j)$.

We will be using the Greek letters $\alpha, \beta, \gamma, \xi, \ldots$ for ordinals and, in particular, for (infinite) cardinals the (Greek) letters $\kappa, \lambda, \mu, \ldots$. The letter ω is reserved for the least inductive set or, in other words, the set of finite ordinals. The class of ordinals is denoted by On and for a limit ordinal α, $cf(\alpha)$ stands for the cofinality of α. Since we are working with the axiom of foundation, if x is a set, $rank(x)$ stands for the rank of x in the cumulative hierarchy of sets. Moreover, the transitive closure of a set x is denoted by $trcl(x)$ and, for a cardinal κ, H_κ is the set of sets hereditarily of cardinality less than κ. For any sets x and y, ${}^y x$ denotes the set of functions from y to x. For a function f and a set x, $f``x$ denotes the image of x under f and $f \restriction x$ the restriction of the function f to x. Furthermore, if κ is a cardinal and x a set, $[x]^{<\kappa}$ denotes the set of all subsets of x of cardinality less than κ.[1]

Now, recall that a *filter* F over a non-empty set S is a subset of $\mathcal{P}(S)$ such that the following conditions hold:

- $\emptyset \notin F$ and $S \in F$.
- If $X, Y \in F$, then $X \cap Y \in F$.
- If $X \in F$ and $X \subseteq Y \subseteq S$, then $Y \in F$.

Moreover, if there exists a non-empty set $X_0 \subseteq S$ such that

$$F = \{X \subseteq S : X_0 \subseteq X\},$$

then, the filter is called *principal* (and if there is not such a X_0, it is called *nonprincipal*). By convention, from now on, whenever we mention ultrafilters, we mean nonprincipal ultrafilters. Additionally, F is an *ultrafilter* if, for every $X \subseteq S$, it holds that either $X \in F$ or $S \setminus X \in F$. For κ a regular cardinal and U an ultrafilter over some set S, we say that U is *κ-complete* if it is closed under intersections of collections of less than κ many sets. [2] Furthermore, if U is over κ and is closed under diagonal intersections of κ many sets, then U is called *normal*.

A tool we will be using is that of ultrapowers (of V). We denote the ultrapower of V with respect to an ultrafilter U over a non-empty set S as $Ult(V, U)$ and its elements as $(f)_U$ (for every function $f : S \to V$). Moreover, if U is (at least) ω_1-complete, then $Ult(V, U)$ is well-founded and thus, its transitive collapse, denoted by M_U, exists. In addition, the elements of M_U are denoted by $[f]_U$, that is,

$$[f]_U = \pi_U((f)_U),$$

where $f : S \to V$ and $\pi_U : Ult(V, U) \to M_U$ is the Mostowski isomorphism. The notation $j_U : V \prec M_U \cong Ult(V, U)$ stands for the usual canonical embedding into the transitive collapse of the ultrapower which, for every $x \in V$, is defined as $j_U(x) = [c_x]_U$, where $c_x : S \to \{x\}$. Of course, when it is clear from context, we will drop the subscripts.

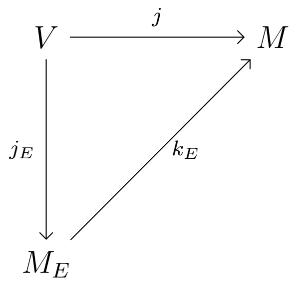
Lastly, suppose that $j : V \prec M$ is an elementary embedding into some transitive M with $cp(j) = \kappa$ and let $\lambda > \kappa$. Then, we denote by $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$ the (κ, λ)-extender derived from j, where for every $a \in [\lambda]^{<\omega}$, E_a is a κ-complete ultrafilter over $[\zeta]^{|a|}$, for some appropriate $\zeta \geq \kappa$, defined as: for every $X \subseteq [\zeta]^a$,

$$X \in E_a \Leftrightarrow a \in j(X).$$

[1] Instead of $[x]^{<\kappa}$, one might have seen the alternative notation $\mathcal{P}_\kappa(x)$.

[2] Note that every ultrafilter is, trivially, ω-complete.

Moreover, M_E stands for the transitive collapse of the direct limit that is derived from E and its elements are denoted by $[a, [f]]$, where $a \in [\lambda]^{<\omega}$ and $f : [\zeta]^{|a|} \to V$. We let $j_E : V \prec M_E$ is the extender embedding which, for every $x \in V$, is defined as $j_E(x) = [a, [c^a_x]]$, for some (any) $a \in [\lambda]^{<\omega}$, where $c^a_x : [\zeta]^{|a|} \to \{x\}$. Finally, recall the third factor elementary embedding $k_E : M_E \prec M$ defined as $k_E([a, [f]]) = j(f)(a)$, for every $a \in [\lambda]^{<\omega}$ and $f : [\zeta]^{|a|} \to V$. In a more schematic point of view, we have the following diagram.



Once again, when it is clear from context, we drop the subscripts (and superscripts).

We now move forward and present two of the most foundamental large cardinal notions.

2.2 Large cardinals

2.2.1 Inaccesibility

As the name suggests, the concept of an inaccessible cardinal wants to capture the notion of a cardinal that, intuitively, can not be "reached from below" and is thus "too large to exist". Of course, this is quite vague (after all this is how intuition works) but, in this subsection, we will try to make it clear.

Definition 2.1. A cardinal κ is *(strongly) inaccessible* if it is uncountable, regular and a strong limit.

The word strongly in the parethesis above is used in order to not confuse this definition with the notion of a *weakly* inaccessible cardinal (which we will not be discussing here). With that being said, from now on, whenever we mention an inaccessible cardinal we mean strongly inaccessible.

A crucial property of inaccessible cardinals is the following.

Proposition 2.2. If κ is inaccessible, then $V_\kappa = H_\kappa$. In addition, we have that $|V_\kappa| = |H_\kappa| = \kappa$.

Proof. First, we show that $H_\kappa \subseteq V_\kappa$, which in fact is true for any infinite cardinal. Let $x \in H_\kappa$ and S be the set $\{rank(y) : y \in trcl(x)\}$. We claim that S is an ordinal. Let γ be the least ordinal not in S. Now, if $\gamma = S$, we are obviously done. On the other hand, if $\gamma \neq S$, we have that $\gamma \subsetneq S$ and so, let β be the least element of S greater than γ and fix $y \in trcl(x)$ with $rank(y) = \beta$. Then, we have that $rank(y) = \sup\{rank(z) + 1 : z \in y\}$ which, by the transitivity of $trcl(x)$, is less than or equal to γ; a contradiction. Hence, we have that $S = \gamma$ (it is indeed an ordinal) and since $|trcl(x)| < \kappa$, it follows that $\gamma < \kappa$. Lastly, since $x \subseteq trcl(x) \subseteq V_\gamma$, we get that $x \in V_\kappa$.

For the other direction, it is easily shown by induction that for every $\alpha < \kappa$, we have that $|V_\alpha| < \kappa$, from which it follows that $|V_\kappa| = \kappa$. Now, if $x \in V_\kappa$, since κ is a limit ordinal, we have that $x \in V_\alpha$ for some $\alpha < \kappa$. Moreover, we have that $trcl(x) \subseteq V_\alpha$ which, according to the above induction, yields that $|trcl(x)| < \kappa$, that is, $x \in H_\kappa$. □

Our first example of a model of ZFC is that of V_κ, for κ an inaccessible.

Theorem 2.3. If κ is inaccessible, then V_κ is a model of ZFC.

Proof. Since κ is a limit ordinal greater than ω, it is easily shown that all the axioms of ZFC except Replacement hold in V_κ. Now, if $x \in V_\kappa$ and $F : x \to V_\kappa$ is a function, from the previous proposition, we have that $|F"x| \leq |x| < \kappa$. Moreover, obviously $F"x \subseteq V_\kappa$ and, since κ is regular, we have that, for some $\alpha < \kappa$, $\{rank(y) : y \in F"x\} \subseteq \alpha$. This in turn yields that $F"x \in V_{\alpha+1} \subseteq V_\kappa$ and thus, V_κ satisfies Replacement as well. □

We immediately get the following relative consistency result.

Corollary 2.4. The following implication holds:

$$Con(ZFC) \Rightarrow Con(ZFC + \neg\exists\kappa(\text{“}\kappa \text{ is inaccessible”})).$$

Proof. First, for simplicity, let $S = ZFC + \neg\exists\kappa(\text{“}\kappa \text{ is inaccessible”})$. We prove the contrapositive: supposing that $Con(S)$ does not hold, we will show that $Con(ZFC)$ does not hold either. So, if $Con(S)$ does not hold, then, recalling that proofs are *finite* objects, there is a *finite* list of axioms of S that proves a contradiction, i.e., there are $\phi_0, \phi_1, \ldots, \phi_n$, which belong to S, such that $\phi_0, \phi_1, \ldots, \phi_n \vdash \psi \land \neg\psi$, for some formula ψ. Moreover, we can assume that the formula $\neg\exists\kappa(\text{“}\kappa \text{ is inaccessible”})$ is one of these ϕ_i, since otherwise we would immediately get that $Con(ZFC)$ does not hold. So, without loss of generality, let ϕ_0 be the formula $\neg\exists\kappa$ (“κ is inaccessible”).

Now, it is easy to see that the inconsistency of S implies that ZFC proves the existence of an inaccessible cardinal. Hence, the existence of the least inaccessible cardinal κ is also provable from ZFC and thus, by (a formal translation of) the previous theorem, we have that

$$ZFC \vdash (\phi_0 \land \ldots \land \phi_n)^{V_\kappa}.$$

But this in turn implies that $ZFC \vdash (\psi \land \neg\psi)^{V_\kappa}$, that is, $Con(ZFC)$ does not hold. □

Remark. The previous corollary is a relative consistency statement; an implication in the meta-theory that relates the consistency of two formal theories. In particular, as the above proof indicates, given an inconsistency of the theory in the consequent of the implication, we get an inconsistency of the theory in the antecedent. This is done in a completely finitistic way which does not involve any infinite (set-theoretic) notion in the meta-theory. This is the only case were we have been that formal, since meta-mathematical issues is not the main interest of this thesis. However, in the relative consistency statements that are to come, the reader should bear in his mind this remark.

We now proceed to the case of measurables, establishing that way the basis for all the other large cardinal notions that we will be examining.

2.2.2 Measurability

Measurable cardinals were introduced by Ulam in 1930's and are of paramount importance for the theory of large cardinals. Their origins are measure-theoretic and, in the course of time, they have found a plethora of applications in many areas, from measure theory and its branches, to group theory and algebra in general. Now, for our purposes, measurable cardinals will mainly play an introductory role for the interconnection between large cardinals, ultrapowers and elementary embeddings; providing us this way with useful information, and the general concepts, that we will be using later on. Here is the definition.

Definition 2.5. An uncountable cardinal κ is called *measurable* if there is a (nonprincipal) κ-complete ultrafilter over κ.

Note that if κ is measurable and U a witnessing κ-complete ultrafilter, then, κ is a regular cardinal since, otherwise, if it was singular it would be the union of fewer than κ "small" sets and, by κ-completeness, this would yield that U is a principal ultrafilter; a contradiction. Moreover, κ is a strong limit: suppose towards a contradiction that there exists a $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. In other words, there is an injective function $f : \kappa \to {}^\lambda 2$. Now, since U is an ultrafilter over κ, for each $\alpha < \lambda$, there is an $i_\alpha \in \{0, 1\}$ such that $X_\alpha = \{\xi < \kappa : f(\xi)(\alpha) = i_\alpha\} \in U$. Furthermore, by κ-completeness, it follows that $\bigcap_{\alpha<\lambda} X_\alpha \in U$ and thus, for every $\xi \in \bigcap_{\alpha<\lambda} X_\alpha$, we have that $f(\xi)(\alpha) = i_\alpha$ for every $\alpha < \lambda$. Hence, X is a singleton, which once again contradicts the nonprincipality of U. Groupping these two facts, we get the following corollary.

Corollary 2.6. If κ is measurable, then κ is an inaccessible cardinal.

Now, if U is a κ-complete ultrafilter over a measurable cardinal κ, then we can construct the ultrapower of V with respect to U. Moreover, since U is ω_1-complete, $Ult(V, U)$ is well-founded and thus, we can take its transitive collapse M_U, as well as define the canonical embedding $j_U : V \prec M_U \cong Ult(V, U)$. A question now is what kind of properties does the critical point of j_U have.

Theorem 2.7. Let κ be a measurable cardinal and U a κ-complete ultrafilter over κ. If $j : V \prec M_U \cong Ult(V, U)$ is the corresponding embedding, then $cp(j) = \kappa$.

Proof. First, we claim that, for every $\alpha < \kappa$, we have that $j(\alpha) = \alpha$. For, suppose, towards a contradiction, that $\alpha < \kappa$ is the least ordinal such that $j(\alpha) > \alpha$. Then, since $\alpha \in M_U$, α

is equal to some $[f]$, where $f : \kappa \to V$. Moreover, $\alpha < j(\alpha)$ implies that $\{\xi < \kappa : f(\xi) < \alpha\} \in U$ and, by κ-completeness, it follows that $\{\xi < \kappa : f(\xi) = \beta\} \in U$, for some $\beta < \alpha$. But this means that $[f] = j(\beta)$ and since, by the minimality of α, $j(\beta) = \beta$, we have that $\alpha = [f] = \beta$; a contradiction.

Now, note that, for every $\alpha < \kappa$, U contains the tail set $C_\alpha = \{\xi < \kappa : \alpha < \xi\}$, which implies that, for every $\alpha < \kappa$, we have that $\alpha = j(\alpha) < [id] < j(\kappa)$, where $id : \kappa \to \kappa$ is the identity function. Hence, we have that $\kappa \leq [id] < j(\kappa)$ and thus, $\kappa = cp(j)$. □

In the opposite direction, consider the following argument. Suppose that $j : V \prec M$ is an elementary embedding with critical point $cp(j) = \gamma$, for some ordinal γ. Then, by basic absoluteness results, it is easy to see that $\gamma > \omega$. Furthermore, let U be defined as follows: for every $X \subseteq \gamma$,

$$X \in U \Leftrightarrow \gamma \in j(X).$$

Now, a standard fact is that U is a γ-complete ultrafilter (in fact, normal) over γ. However, to verify this fact is quite technical, and a complete proof of it here would distract us from what we want to indicate. So, we leave the details for the interested reader. The important thing is that we have just shown the following, alternative, characterization of measurability.

Corollary 2.8. A cardinal κ is measurable if and only if it is the critical point of some elementary embedding $j : V \prec M$.

Note that the above characterization of measurable cardinals uses quantification over proper classes which of course is not formalizable in first-order ZFC.

Now, given an elementary embedding j of the form above, the aforementioned observation indicates a way to construct from j a $cp(j)$-complete ultrafilter U over $cp(j)$. Additionally, using U we can build the ultrapower of V and then, derive another elementary embedding j_U; the canonical embedding into the transitive collapse M_U. In some cases this could turn up being very useful, since M_U is definable from U and, moreover, its structure enjoys some nice properties. Some of these properties are the following.

Proposition 2.9. Suppose U is a κ-complete ultrafilter over $\kappa > \omega$ and $j : V \prec M \cong Ult(V, U)$ is the corresponding embedding. Then, the following hold:

1. For every $x \in V_\kappa$, we have that $j(x) = x$, that is, $j \restriction V_\kappa$ is the identity.
2. For every $X \subseteq V_\kappa$, we have that $j(X) \cap V_\kappa = X$.
3. $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$.
4. ${}^\kappa M \subseteq M$ and ${}^{\kappa^+} M \nsubseteq M$.
5. $U \notin M$.

Proof. For 1, let x be of least rank such that $j(x) \neq x$ and suppose that $rank(x) = \gamma$; we will show that $\gamma \geq \kappa$. If $y \in x$, then, by elementarity we have that $j(y) \in j(x)$ and, by the minimality of the rank of x, we have that $j(y) = y$. Thus, $x \subseteq j(x)$ and, since $j(x) \neq x$, there is a $z \in j(x) \setminus x$. Now, we want to show that $rank(j(x)) > \gamma$ since, then, by basic absoluteness results, we would have that $j(\gamma) = rank(j(x)) > \gamma$ and, from Theorem 2.7, we would get that $cp(j) = \kappa$ and thus, that $\gamma \geq \kappa$. We prove this by contradiction. First, observe that $rank(j(x)) \geq \gamma$. Next, suppose, towards a contradiction, that $rank(j(x)) = \gamma$. Then, it follows that $j(z) = z$ and so, $z \in x$, which of course is a contradiction.

For 2, suppose that $X \subseteq V_\kappa$. If $z \in j(X) \cap V_\kappa$, then, $j(z) = z$ and thus $j(z) \in j(X)$, which by elementarity implies that $z \in X$. If on the other hand $z \in X$, then, since $X \subseteq V_\kappa$, we have that $j(z) = z$ and, by elementarity, $z \in j(X) \cap V_\kappa$.

For 3, using 2, we have that $\mathcal{P}^M(\kappa) = \mathcal{P}(\kappa)$ and thus, by the definition of the cardinality of a set, it follows that $2^\kappa \leq (2^\kappa)^M$. Moreover, since κ is inaccessible, by elementarity, we have that $M \vDash$ "$j(\kappa)$ is inaccessible" and thus, $M \vDash 2^\kappa < j(\kappa)$, which in turn means that $(2^\kappa)^M < j(\kappa)$. Finally, observe that $j(\kappa) = \{[f] : f \in {}^\kappa\kappa\}$, since, if $f : \kappa \to V$, then $[f] \in [c_\kappa] = j(\kappa)$ if and only if $\{\xi < \kappa : f(\xi) < \kappa\} \in U$ and so, we might as well consider functions only of the form $f : \kappa \to \kappa$. This implies that $j(\kappa) < (2^\kappa)^+$.

For 4, suppose that $\{[f_\alpha] : \alpha < \kappa\} \subseteq M$ and let $g : \kappa \to \kappa$ be such that $[g] = \kappa$. Moreover, for every $\xi < \kappa$, let $H(\xi)$ be that function with domain $g(\xi)$ satisfying $(H(\xi))(\alpha) = f_\alpha(\xi)$, for every $\alpha < g(\xi)$. This means that $[H]$ is a function with domain κ and, for every $\alpha < \kappa$, $[H](\alpha) = [f_\alpha]$. In other words, we have that $H = \langle [f_\alpha] : \alpha < \kappa \rangle$, which shows that M is indeed closed under κ-sequences. As for the second part, the required result follows from the observation that $j"\kappa^+ \notin M$, since otherwise, if $j"\kappa^+ \in M$, then, inside M there would be a cofinal function from $j"\kappa^+$, which has order type κ^+, to $j(\kappa^+)$, contradicting the fact that $M \vDash$ "$j(\kappa^+)$ is regular" (which by elementarity M believes, since every successor cardinal, in particular κ^+, is regular).

Lastly, for 5, note that by 2, we have that $({}^\kappa\kappa)^M = {}^\kappa\kappa$. Now, for the sake of contradiction, suppose that $U \in M$. Then, inside M, for every $f \in {}^\kappa\kappa$ we can construct the equivalence class $[f]$ and, moreover, note that the surjection $g : {}^\kappa\kappa \to j(\kappa)$, where $f \mapsto [f]$, would also belong to M. But this means that $M \vDash |j(\kappa)| \leq 2^\kappa$ and thus, it would contradict the inaccessibility of $j(\kappa)$ inside M. □

The above proposition provides us with a lot of information about the structure of the transitive collapse of the ultrapower and so, before ending this subsection, let us first make a few notes.

First, note that (1) above implies that $V_\kappa^M = V_\kappa$, that is, up to κ, V and M "aggree" on how the universe looks like. Moreover, (2) strengthens this a bit more since it implies that $V_{\kappa+1}^M = V_{\kappa+1}$ and $\kappa^{+M} = \kappa^+$, the second one being a consequence of M containing every well-ordering of κ. Additionally, (3) implies that, even though M "thinks" that $j(\kappa)$ is an inaccessible cardinal, the truth (according to V) is that $j(\kappa)$ is *not* even a cardinal, since its place is somewhere in between 2^κ and $(2^\kappa)^+$.

These were the basic concepts regarding inaccessible and measurable cardinals that we will, silently, be using in the subsequent sections. We now climb the large cardinal hierarchy and delve into the realm of, what is sometimes called, *very* large cardinals.

3. VERY LARGE CARDINALS

The motivation behind the (very) large cardinal notions we will be presenting is based on the elementary embedding characterization of measurability and the closure properties of the corresponding target model. In particular, we now consider elementary embeddings from V into some M that contain much more information than those that "simply" witness the measurability of a cardinal. As we will shortly see, this is a general pattern which leads to large cardinal axioms of increasing consistency strength.

Our first example is that of a superstrong cardinal.

3.1 Superstrongness

Apart from having an elementary embedding characterization, superstrong cardinals constitute an elegant, and simple, example of large cardinals that can be defined via extenders. Hence, we seize the opportunity to briefly discuss some extender-related issues. Moreover, let us also mention that, for our purposes superstrongness will play a crucial role when we will reach the point of discussing the relation between $C^{(n)}$-extendible and $C^{(n)}$-supercompact cardinals (from Definition 4.12 and below).

Definition 3.1. A cardinal κ is superstrong if there exists an elementary embedding $j : V \prec M$ with M transitive, $cp(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$.

First, observe that, if κ is superstrong, then it is also measurable. Secondly, just like the alternative definition of measurability via elementary embeddings, the above definition is informal in the sense that it too requires quantification over proper classes. However, the next proposition suggests that there is an equivalent definition of "combinatorial nature" which is formalizable in first-order ZFC.

Proposition 3.2. A cardinal κ is superstrong if and only if for some $\lambda > \kappa$ there is a (κ, λ)-extender E such that $V_{j_E(\kappa)} \subseteq M_E$.

Proof. Note that, the converse direction is trivial. For the forward direction, let κ be a cardinal and $j : V \prec M$ an elementary embedding witnessing the superstrongness of κ. Moreover, let E be the $(\kappa, j(\kappa))$-extender derived from j and let $j_E : V \prec M_E$ be the corresponding extender embedding, where, from [17, Prop. 2.5(i)], we have that $j_E(\kappa) = j(\kappa)$. Furthermore, since $M \models$ "$j(\kappa)$ is inaccessible", we have that $M \models |V_{j(\kappa)}| = j(\kappa)$ and thus, from [17, Prop. 2.5(iii)], it follows that $V^M_{j(\kappa)} = V^{M_E}_{j(\kappa)}$. Lastly, by the superstrongness of j, we have that $V_{j(\kappa)} = V^M_{j(\kappa)}$ and thus, summing up, it follows that $V_{j_E(\kappa)} = V_{j(\kappa)} = V^{M_E}_{j(\kappa)} \subseteq M_E$. □

With this in mind, and recalling that verifying if E is an extender is something that can be checked in V_α for some sufficiently large ordinal α, note that the defining complexity of superstrongness is Σ_2.

Now, the proof of the following proposition is full of useful ideas that will be used later on in our study.

Proposition 3.3. Let κ be a cardinal and suppose that $j : V_{\kappa+1} \prec V_{j(\kappa)+1}$ is an elementary embedding with $cp(j) = \kappa$. Then, κ is superstrong and there is a normal ultrafilter U over κ such that

$$\{\alpha < \kappa : \text{“}\alpha \text{ is superstrong”}\} \in U.$$

Proof. The idea is standard; we will prove that $V_{j(\kappa)+1} \vDash$ “κ is superstrong”, define the normal ultrafilter U derived from j and then, using the usual reflection argument on U, we will get what we want. However, for the first part, we first have to deal with some “extender issues”.

Let κ be a cardinal and $j : V_{\kappa+1} \prec V_{j(\kappa)+1}$ an elementary embedding with $cp(j) = \kappa$. Note that, for every $n \in \omega$, we have that $\mathcal{P}([\kappa]^n) \subseteq V_{\kappa+1}$ and that $[j(\kappa)]^{<\omega} \subseteq V_{j(\kappa)}$. So, even though in this particular case we are not dealing with an elementary embedding between class models, we can define, for each $a \in [j(\kappa)]^{<\omega}$, the ultrafilters E_a and derive the usual $(\kappa, j(\kappa))$-extender E from j. More precisely, we define $E = \langle E_a : a \in [j(\kappa)]^{<\omega}\rangle$ as follows: for every $a \in [j(\kappa)]^{<\omega}$ and for every $X \subseteq [\kappa]^{|a|}$,

$$X \in E_a \Leftrightarrow a \in j(X).$$

Now, since j is between sets (and not class models), we have to check that E is in fact a $(\kappa, j(\kappa))$-extender. Fortunately, probably by first turning back to the definition of an extender, it is easy to see that all the relevant sets are present, as well as that all the conditions of the definition of an extender are indeed satisfied. Moreover, $E \in V_{j(\kappa)+1}$ and thus, it can be checked inside $V_{j(\kappa)+1}$ that E is indeed a $(\kappa, j(\kappa))$-extender. Hence, we have that $V_{j(\kappa)+1} \vDash$ “E is a $(\kappa, j(\kappa))$-extender” which, crucialy, is correct in V.

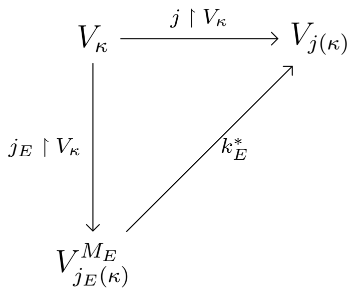
We can now define the extender elementary embedding $j_E : V \prec M_E$. Moreover, we can define $k^*_E : V^{M_E}_{j_E(\kappa)} \to V_{j(\kappa)}$ by letting

$$k^*_E([a, [f]]) = j(f)(a),$$

for all $[a, [f]] \in V^{M_E}_{j_E(\kappa)}$ with $a \in [j(\kappa)]^{<\omega}$ and appropriate $f : [\kappa]^{|a|} \to V$. Of course, we now have to check that the above definition makes sence. To see this, observe that $V^{M_E}_{j_E(\kappa)} = j_E(V_\kappa)$ and $j_E(V_\kappa)$ is by definition equal to $[a, [c^a_{V_\kappa}]]$, for some (any) $a \in [j(\kappa)]^{<\omega}$. Hence, if $[a, [f]] \in V^{M_E}_{j_E(\kappa)}$, we have that $[a, [f]] \in [a, [c^a_{V_\kappa}]]$. In other words, we have that $f(s) \in V_\kappa$ for almost all $s \in [\kappa]^{|a|}$, i.e., for every $a \in [j(\kappa)]^{<\omega}$, the set $\{s \in [\kappa]^{|a|} : f(s) \in V_\kappa\}$ belongs to E_a. With that in mind, we may assume that the functions f are of the form $f : [\kappa]^{|a|} \to V_\kappa$ and thus, that each such function belongs to $V_{\kappa+1}$, which in turn yields that, for every $a \in [j(\kappa)]^{<\omega}$, $j(f)(a)$ belongs to $V_{j(\kappa)}$. Lastly, recall that we have the following equivalences:

$$[a, [f]] = [b, [g]] \Leftrightarrow j(f)(a) = j(g)(b)$$
$$[a, [f]] \in [b, [g]] \Leftrightarrow j(f)(a) \in j(g)(b),$$

which concludes the proof of k^*_E being a well-defined map and also verifies that k^*_E is a $\{\in\}$-embedding. Hence, in a more schematic point of view, we have the following commutative diagram:



Moreover, we claim that k_E^* is in fact the identity map. First, since k_E^* is a embedding, it is an injection. Furthermore, since κ is inaccessible, we have that $|V_\kappa| = \kappa$. So, let $g : [\kappa]^1 \to V_\kappa$ be a bijection, which obviously belongs in $V_{\kappa+1}$. By elementarity, we have that $j(g) : [j(\kappa)]^1 \to V_{j(\kappa)}$ is also a bijection and that $j(g) \in V_{j(\kappa)+1}$. In other words, for every $x \in V_{j(\kappa)}$, there is some $\gamma < j(\kappa)$ such that $x = j(g)(\{\gamma\})$. Now, by the definition of k_E^*, this means that, for every $x \in V_{j(\kappa)}$, there is some $\gamma < j(\kappa)$ such that $k_E^*([\{\gamma\}, [g]]) = x$, that is, k_E^* is also a surjection. Putting it together, we have that k_E^* is a bijection between transitive sets and thus, it is the identity function.

Hence, we have that $V_{j_E(\kappa)}^{M_E} = V_{j(\kappa)}$, from which it follows that $V_{j(\kappa)} \subseteq M_E$ and $j_E(\kappa) = j(\kappa)$. Therefore, from Proposition 3.2, we get that κ is a superstrong cardinal. Now, in order to conclude the proof, we have to show that the superstrongness of κ is witnessed by $V_{j(\kappa)+1}$.

The key observation is that $V_{j(\kappa)+1}$ is large enough to compute correctly $j_E(\kappa)$ and $j_E(V_\kappa)$. More precisely, we claim that $(j_E)^{V_{j(\kappa)+1}}(\kappa) = j_E(\kappa)$ and $(j_E)^{V_{j(\kappa)+1}}(V_\kappa) = j_E(V_\kappa)$. For the former, recall that the order type of $j_E(\kappa)$ is the order type of the set

$$\{[a, [f]] : a \in [j(\kappa)]^{<\omega}, f : [\kappa]^{|a|} \to \kappa\}$$

and since $[j(\kappa)]^{<\omega} \in V_{j(\kappa)+1}$ and all functions of the form $f : [\kappa]^{|a|} \to \kappa$ belong to $V_{\kappa+1} \subseteq V_{j(\kappa)+1}$, we have that $(j_E)^{V_{j(\kappa)+1}}(\kappa) = j_E(\kappa) = j(\kappa)$. For the latter, we have already mentioned that $j_E(V_\kappa) = V_{j_E(\kappa)}^{M_E}$ and that $V_{j_E(\kappa)}^{M_E} = V_{j(\kappa)} \subseteq (M_E)^{V_{j(\kappa)+1}}$. Hence, we have that $V_{j(\kappa)+1} \models V_{j(\kappa)} \subseteq M_E$. It now follows that

$$V_{j(\kappa)+1} \models \text{“}\kappa \text{ is superstrong”},$$

and, if we define from j the normal ultrafiler U (recall the paragraph above Corollary 2.8) where, for every $X \subseteq \kappa$, we have that

$$X \in U \Leftrightarrow \kappa \in j(X),$$

then it follows that $\{\alpha < \kappa : V_{\kappa+1} \models \text{“}\alpha \text{ is superstrong”}\} \in U$, which in turn yields that $\{\alpha < \kappa : \text{“}\alpha \text{ is superstrong”}\}$ belongs to U. □

At this point, let us mention that in Section 2.3 we will introduce the notion of extendible cardinals (see Definition 3.9). In knowledge of that large cardinal notion, the elementary embedding in the previous proposition, i.e., $j : V_{\kappa+1} \prec V_{j(\kappa)+1}$ with $cp(j) = \kappa$, witnesses the $\kappa+1$-extendibility of κ. In other words, κ above is a $\kappa+1$-extendible cardinal and, moreover, the preceding proposition can be understood in these terms.

Moving forward, we next present the case of supercompactness.

3.2 Supercompactness

Supercompact cardinals are a highly significant concept in large cardinal theory. They were originally introduced by Solovay and Reinhardt and since then have been explored by many others. Their strong reflection properties have led in a number of profound implications in many areas of set theory, as well as other parts of mathematics.

Definition 3.4. A cardinal κ is γ-*supercompact*, for some $\gamma \geq \kappa$, if there is an elementary embedding $j : V \prec M$ with M transitive, $cp(j) = \kappa$, $\gamma < j(\kappa)$ and ${}^{\gamma}M \subseteq M$. Moreover, κ is *supercompact* if it is γ-supercompact for every $\gamma \geq \kappa$.

Observe that, a cardinal κ is κ-supercompact if and only if κ is measurable. Furthermore, as the following proposition suggests, the notion of supercompactness transcends superstrongness, in the sense that, if κ is the least superstrong cardinal, then it can not be 2^κ-supercompact, let alone fully supercompact.

Proposition 3.5. If κ is 2^κ-supercompact, then, there is a κ-complete ultrafilter U over κ such that

$$\{\alpha < \kappa : \text{“}\alpha \text{ is superstrong”}\} \in U.$$

Proof. Consider an elementary embedding $j : V \prec M$ with $cp(j) = \kappa$, $j(\kappa) > 2^\kappa$ and ${}^{2^\kappa}M \subseteq M$, i.e., a witness of the 2^κ-supercompactness of κ. Moreover, let $j^* = j \upharpoonright V_{\kappa+1}$ where $j \upharpoonright V_{\kappa+1} : V_{\kappa+1} \to V^M_{j(\kappa)+1}$. Now, since $V^M_{\kappa+1} = V_{\kappa+1}$, by the closure of M under 2^κ-sequences, we have that $j^* \in M$. This in turn implies that

$$M \vDash \text{“}j^* : V_{\kappa+1} \to V_{j(\kappa)+1} \text{ is an elementary embedding with } cp(j^*) = \kappa\text{”},$$

which in its own turn, by Proposition 3.3, implies that $M \vDash$ “κ is superstrong”. Furthermore, observe that, just like in the proof of Proposition 3.3, we can define the normal ultrafilter U and, by similar arguments, we get that $\{\alpha < \kappa : \text{“}\alpha \text{ is superstrong”}\}$ belongs to U. □

Observe that, given an elementary embedding j with $cp(j) = \kappa$, we have used quite some times the following normal ultrafilter over the cardinal κ defined as: for every $X \subseteq \kappa$,

$$X \in U \Leftrightarrow \kappa \in j(X).$$

Its definition together with the arguments we used, conceal a general strategy when someone wants to show that the existence of some large cardinal axiom has some consequences in the universe below it. Consider the following, general, argument. Suppose that $j : V \prec M$ is an elementary embedding from V into M with $cp(j) = \kappa$, and suppose that we want to prove that the set $\{\alpha < \kappa : \phi(\alpha)\}$, where $\phi(x)$ is a first-order formula in the language of set theory, is unbounded below κ. Then, observe that, if we derive from j the usual normal ultrafilter U, by its definition, we only have to show that $M \vDash \phi(\kappa)$. This is a quite useful “trick” that we will be using later on and, for future reference, we name it as *the reflection argument of (the measure)* U.

Moving forward, we now show that supercompact cardinals are Σ_2-correct in V.

Theorem 3.6. If κ is a supercompact cardinal, then V_κ is a Σ_2-elementary substructure of V, that is, $V_\kappa \prec_2 V$.

Proof. Suppose κ is a supercompact cardinal, $a \in V_\kappa$ and $\phi(x)$ is a Σ_2 formula, i.e., a formula of the form $\exists y \psi(x,y)$ where $\psi(x,y)$ is Π_1. We will show that $V_\kappa \vDash \phi(a)$ if and only if $\phi(a)$ holds.

First, by a well-known theorem of Lévy, we have that for every cardinal $\lambda > \omega$, $H_\lambda \prec_1 V$ and so, in our case, we have that $H_\kappa \prec_1 V$. Moreover, since κ is inaccessible, we have that $V_\kappa = H_\kappa$ and thus, that $V_\kappa \prec_1 V$. Hence, if $V_\kappa \vDash \phi(a)$, that is, $V_\kappa \vDash \psi(a,b)$ for some $b \in V_\kappa$, then, since $\psi(a,b)$ is Π_1, we have that $\psi(a,b)$ holds, i.e. $\phi(a)$ holds.

On the other hand, if $\psi(a,b)$ holds for some $b \in V$, let γ be an ordinal greater than the rank of b. Then, since κ is supercompact, it is in particular $|V_\gamma|$-supercompact. So, let $j : V \prec M$ be a witness of the $|V_\gamma|$-supercompactness of κ and observe that, by the closure of M, we have that $b \in M$ and, in particular, that $b \in V^M_{j(\kappa)}$. Now, since $\psi(a,b)$ is Π_1, by downwards absoluteness we have that $M \vDash \psi(a,b)$ and, moreover, that $M \vDash \big(V_{j(\kappa)} \vDash \psi(a,b)\big)$. In other words, we have that $M \vDash \big(V_{j(\kappa)} \vDash \phi(a)\big)$ and, since $j(a) = a$, by elementarity it follows that $V_\kappa \vDash \phi(a)$. □

Let us now briefly comment upon the formalization of the notion of supercompactness. Once again, note that the definition of supercompactness is *not* formalizable in first-order ZFC, but, nevertheless, by considering ultrafilters that contain more information than those in the case of measurability, one can aquire the following alternative defition that is formalizable in first-order ZFC: a cardinal κ is γ-supercompact, for some $\gamma \geq \kappa$, if and only if there is a *normal* and *fine*[1] ultrafilter over $[\gamma]^{<\kappa}$.

However, the path leading to this characterization is quite technical and it would distract us from our main purpose. Hence, for further details, we advise the interested reader to consult [11, Sec. 22].

With that being said, observe that if κ is a cardinal and $\gamma \geq \kappa$, then, using the above characterization of supercompactness, one can show that the statement "κ is γ-supercompact" is Δ_2, which in turn yields that the statement "κ is supercompact" is Π_2. Furthermore, this is optimal, since if it was of lower complexity, say Π_1, then the statement "there exists a supercompact cardinal" would be Σ_2. Now, if we take the least supercompact cardinal κ, then by Theorem 3.6, V_κ would satisfy the existence of a supercompact cardinal, which clearly is a contradiction.

Suppose now that κ is supercompact and $\lambda > \kappa$ is a limit ordinal. Then, we claim that $V_\lambda \vDash$ "κ is supercompact". To see this, we use the aforementioned alternative definition of supercompactness: for every $\gamma < \lambda$, let U_γ be a normal and fine ultrafilter over $[\gamma]^{<\kappa}$. Then, since λ is a limit ordinal, all these ultrafilters belong to V_λ and so, it follows that $V_\lambda \vDash$ "κ is supercompact".

Another useful proposition is the following.

[1] For the definition of a normal fine ultrafilter see [11, Sec. 22]

Proposition 3.7. Let κ and λ be cardinals. Moreover, suppose that κ is γ-supercompact for every $\kappa \leq \gamma < \lambda$ and that λ is supercompact. Then κ is supercompact.

Proof. Since λ is supercompact it is an inaccessible cardinal and in particular a limit ordinal. So, we have that $V_\lambda \vDash$ "κ is supercompact". Now, by Theorem 3.6 this is true in V, or in other words, κ is indeed supercompact. □

Furthermore, we have the following relative consistency statement.

Corollary 3.8. The following holds:

$$Con\big(ZFC + \exists\kappa(\text{“}\kappa \text{ is supercompact”})\big)$$
$$\Downarrow$$
$$Con\big(ZFC + \exists\kappa(\text{“}\kappa \text{ is supercompact”}) + \neg\exists\lambda(\lambda > \kappa \wedge \text{“}\lambda \text{ is inaccessible”})\big)$$

Proof. First, recalling the remark after Corollary 2.4, let us mention that the following proof is a bit informal, but, nevertheless, one can easily fill in the missing details.

Suppose that κ is a supercompact cardinal. There are two cases: either there are not any inaccessible cardinals greater than κ or there is at least one. If it is the former case, then we are done. If it is the latter, let λ be the least inaccessible cardinal above κ. Then, by Theorem 2.3, the previous observation and by the minimality of λ, it follows that

$$V_\lambda \vDash ZFC + \exists\kappa(\text{“}\kappa \text{ is supercompact”}) + \neg\exists\lambda(\lambda > \kappa \wedge \text{“}\lambda \text{ is inaccessible”})$$

which is what we wanted. □

So, according to the preceding corollary, the existence of a supercompact cardinal κ does not imply the existence of any other large cardinal stronger than an inaccessible above κ; a quite interesting fact.

As a last example, we present extendibles, a large cardinal notion which will be particulary useful for our study.

3.3 Extendibility

An extendible cardinal is a (very) large cardinal notion that is, as we will see, closely connected to that of supercompactness. Once again, we will be dealing with elementary embeddings but, this time, the elementary embeddings will be between sets, and not class models. The main definition is the following.

Definition 3.9. A cardinal κ is λ-*extendible*, for some $\lambda > \kappa$, if there exists a $\mu > \lambda$ and an elementary embedding $j : V_\lambda \prec V_\mu$ with $cp(j) = \kappa$ and $j(\kappa) > \lambda$. Moreover, κ is *extendible* if it is λ-extendible for every $\lambda > \kappa$.

Note that, if a cardinal κ is $\kappa+1$-extendible and $j : V_{\kappa+1} \prec V_\mu$ is a witnessing elementary embedding, then $\mu = j(\kappa)+1$. Moreover, the following proposition suggests that extendibility is a much stronger notion than that of measurability.

Proposition 3.10. If κ is $\kappa+1$-extendible, then κ is measurable and, moreover, the set of measurable cardinals below it is unbounded in κ.

Proof. To see this, suppose that $j : V_{\kappa+1} \prec V_{j(\kappa)+1}$ witnesses the $\kappa+1$-extendibility of κ. Then, noting that $\mathcal{P}(\kappa) \subseteq V_{\kappa+1}$, we define the usual normal ultrafilter U over κ which belongs to $V_{j(\kappa)+1}$. But this means that, $V_{j(\kappa)+1} \vDash$ "κ is measurable", which of course is correct in V and so, κ is indeed a measurable cardinal. Lastly, by the reflection argument of U, we get that $\{\alpha < \kappa : V_{\kappa+1} \vDash$ "α is measurable"$\} \in U$ and thus, that the set $\{\alpha < \kappa :$ "α is measurable"$\}$ belongs to U. □

Observe also that the set of measurable cardinals is unbounded below $j(\kappa)$: we have that $V_{\kappa+1} \vDash$ "$\{\alpha < \kappa :$ "α is measurable"$\}$ is unbounded below κ" and, by elementarity, $V_{j(\kappa)+1}$ "thinks" that the set $\{\alpha < j(\kappa) :$ "α is measurable"$\}$ is unbounded below $j(\kappa)$, which is easily seen that it is correct in V.

With that being said, note that if κ is extendible, then, there are arbitrarily large measurable cardinals, i.e., measurable cardinals form a proper class. This suggests that, in contrast to the case of supercompactness, extendibility implies the existence of some large cardinals higher in the universe (but, of course, not every large cardinal).

Furthermore, we will later see that extendibility is stronger than supercompactness. More precisely, the special case $n = 1$ of Corollary 4.16 suggests that, if κ is extendible, then it is also supercompact.

As for the defining complexity of extendibility, if κ is a cardinal and $\lambda > \kappa$, then, we have that "κ is λ-extendible" if and only if the following holds:

$$\exists\mu\exists j\big(\text{“}j : V_\lambda \to V_\mu \text{ is elementary ”} \wedge cp(j) = \kappa \wedge j(\kappa) > \lambda\big),$$

which is easily seen that it is Σ_2. Hence, the property of being (fully) extendible is a Π_3 property.

With that being said, we now present one more relative consistency result.

Theorem 3.11. The following holds:

$$Con\big(ZFC + \exists\kappa(\text{“}\kappa \text{ is extendible”})\big)$$
$$\Downarrow$$
$$Con\big(ZFC + \exists\kappa(\text{“}\kappa \text{ is extendible”}) + \neg\exists\lambda(\lambda > \kappa \wedge \text{“}\lambda \text{ is supercompact”})\big)$$

Proof. First, observe that if κ is extendible and $\lambda > \kappa$ is supercompact, then, V_λ "thinks" that κ is an extendible cardinal since, by Theorem 3.6, we have that λ is Σ_2-correct in V and thus, by downwards absoluteness, V_λ believes that κ is extendible.

Suppose now κ is an extendible cardinal and, as usual, consider two cases. If there are no supercompact cardinals above κ, then we are done. If on the other hand there is at least one, let λ be the least supercompact above κ. From the above observation, we have that $V_\lambda \vDash$ "κ is extendible" and, since λ is inaccessible, we also have that $V_\lambda \vDash ZFC$. Hence, it remains to prove that V_λ does not think that there are any supercompact cardinals above κ. Suppose, towards a contradiction, that

$$V_\lambda \vDash \exists \mu > \kappa (\text{"}\mu \text{ is supercompact"}).$$

But then, since λ is Σ_2-correct in V and being supercompact is a Π_2 property, this means that μ is indeed a supercompact cardinal below λ, a contradiction. □

Lastly, in analogy to Theorem 3.6, extendible cardinals have stronger reflection properties than supercompacts.

Theorem 3.12. If κ is extendible, then $V_\kappa \prec_3 V$.

Proof. Let $a \in V_\kappa$ and $\phi(x)$ be a Σ_3 formula, i.e., a formula of the form $\exists y \psi(x, y)$ where $\psi(x, y)$ is Π_2. Now, if $V_\kappa \vDash \phi(a)$, then, as mentioned above κ is supercompact and thus, by Theorem 3.6 and by upwards absoluteness we have that $\phi(a)$ holds. On the other hand, if $\phi(a)$ holds, i.e., $\psi(a, b)$ holds for some b, then let $\lambda, \mu > \kappa$ and let $j : V_\lambda \prec V_\mu$ be a witness of the λ-extendibility of κ, i.e., $cp(j) = \kappa$ and $j(\kappa) > \lambda$. Then, it is easy to see that $V_\kappa \prec V_{j(\kappa)}$ and, moreover, that $j(\kappa)$ is an inaccessible cardinal. Now, since $\psi(a, b)$ is Π_2 and $b \in V_{j(\kappa)}$, by downwards absoluteness, we have that $V_{j(\kappa)} \vDash \phi(a)$ and, by elementarity, it follows that $V_\kappa \vDash \phi(a)$. □

These were some of the basic concepts of large cardinal theory, aiming at putting the reader into perspective. We are now ready to introduce the notion of $C^{(n)}$-cardinals but, first, we end this chapter with a small remark on an important theorem of Kunen, know as *Kunen's inconsistency theorem* [13].

As we have already mentioned, large cardinals form a hierarchy in terms of consistency strength. Additionally, we have seen that the more information an elementary embedding witnessing a large cardinal notion encodes, the stronger the large cardinal axiom is. Putting it together, one may wonder why not consider the strongest such elementary embedding; a nontrivial elementary embedding from V into itself.

In 1971, Kunen proved that there can be no such elementary embedding and thus, provided us with an upper bound in the hierarchy of large cardinal axioms defined using the machinery of elementary embeddings. In other words, the pattern of postulating powerful large cardinal axioms via elementary embeddings has a ceiling; if we want to remain inside the boundaries of consistency, or at least if we hope we are, we can never have the ultimate closure conditions.

However, it should be mentioned that Kunen's proof (as well as all other known proofs of his theorem) is using the Axiom of Choice in an essential way. Hence, a question which arises is, what if we drop the assumption of AC? Is Kunen's inconsistency theorem still provable? With the aim of answering this question, a lot of work has been done in a choiceless framework but, until now, the question remains open.

4. $C^{(n)}$-CARDINALS

Following the "elementary embedding point of view" of large cardinals, we want to enrich this aspect with the concept of set-theoretic reflection. More precisely, if j is an elementary embedding with critical point κ, we want to investigate what happens when, for some $n \geq 1$, $j(\kappa)$ is Σ_n-correct in V. This seems promising since, after all, for an elementary embedding, the stronger the closure and reflection properties of the target model are, the stronger set-theoretic consequences we have.

The content of this chapter, unless otherwise stated, is due to Bagaria [2].

4.1 Prelude to $C^{(n)}$-cardinals

We begin by giving the general definition of $C^{(n)}$-cardinals.

Definition 4.1. For every $n \geq 0$, we let $C^{(n)}$ be the collection of ordinals which are Σ_n-correct in the universe, that is, we let $C^{(n)} = \{\alpha \in On : V_\alpha \prec_n V\}$.

Note that, for every $n \geq 0$, if $\alpha \in C^{(n)}$ and ϕ is a Σ_{n+1} formula with parameters in V_α such that $V_\alpha \vDash \phi$, then ϕ holds (i.e, for every $n \geq 0$, Σ_{n+1} formulas are upwards absolute for $C^{(n)}$-cardinals). Similarly, if ϕ is a Π_{n+1} formula with parameters in V_α and ϕ holds, then $V_\alpha \vDash \phi$ (i.e, for every $n \geq 0$, Π_{n+1} formulas are downwards absolute for $C^{(n)}$-cardinals).

One may wonder, for each particular $n \geq 0$, how does the structure of $C^{(n)}$ looks like. There are two particular cases for which there is an exact characterization; for $n = 0$ and for $n = 1$. For the former, we have that $C^{(0)} = On$, since, for every ordinal α, V_α is transitive and Δ_0 formulas are absolute for transitive models. For the latter, we have that $C^{(1)} = \{\alpha \in On : \text{“}\alpha \text{ is uncountable”} \wedge V_\alpha = H_\alpha\}$, since, if $\alpha \in C^{(1)}$, then for any $\beta < \alpha$, the formula

$$\exists\gamma\exists f\big(\gamma \in On \wedge \text{“}f : \gamma \to V_\beta \text{ is a surjection”}\big)$$

is Σ_1 in the parameter V_β, and so, it must hold in V_α. Now, observe that this implies that α is an uncountable strong limit cardinal and hence, $V_\alpha = H_\alpha$. On the other hand, for α an uncountable ordinal, if $V_\alpha = H_\alpha$, then, by a theorem of Lévy, we have that $V_\alpha \prec_1 V$.

As for the general case, we have the following, simple, but informative fact.

Theorem 4.2. For every $n \geq 0$, $C^{(n)}$ is a club proper class.

Proof. We prove this by induction (in the meta-theory). For the base case, we have already seen that $C^{(0)} = On$. Suppose now that $C^{(n)}$ is a club proper class; we will show that $C^{(n+1)}$ is also a club proper class.

Let $\langle \phi_m : m \in \omega\rangle$ be an enumeration of the Σ_{n+1} formulas of the language of set theory and, for simplicity, suppose they are of the form $\exists x\psi_m(x, y)$, where $\psi_m(x, y)$ is a Π_n formula.

Moreover, for $\alpha \in On$, let λ_α be the next $C^{(n)}$-cardinal above α (such a cardinal exists since, by the induction hypothesis, $C^{(n)}$ is unbounded in On).

Now, fix some ordinal γ. To prove that $C^{(n+1)}$ is unbounded, we will find a $C^{(n+1)}$ cardinal above γ. More precisely, the idea is to find, using the induction hypothesis, a $C^{(n)}$ cardinal greater than γ for which the universe at that stage contains the required existential witnesses for the Σ_{n+1} formulas.

So, we define recursively the following sequence $\langle d_m : m \in \omega\rangle$: for d_0, we check for every $b \in V_\gamma$ if there exists an a such that $\psi_0(a, b)$ holds. If this is the case, for every such $b \in V_\gamma$, we let α_b be the least ordinal in $C^{(n)}$ (above γ) for which there is such a witness $a \in V_{\alpha_b}$ and then set $d_0 = \sup\{\alpha_b : b \in V_\gamma\}$. Otherwise, if for every $b \in V_\gamma$ there is not any witness a, we set d_0 be the least ordinal in $C^{(n)}$ greater than γ. Note that, in both cases, finding such an ordinal that belongs to $C^{(n)}$ is possible by the induction hypothesis. Continuing with the recursive definition, for $m + 1$, we use the same idea. However, now, we have to keep in mind that there might be new witnesses of some formula up to ϕ_{m+1}. Hence, for every $k \leq m + 1$ and for every $b \in V_{d_m}$, we check if there exists an a such that $\psi_k(a, b)$ holds and, if this is the case, we let α_b be the least cardinal in $C^{(n)}$ for which there is such a witness $a \in V_{d_m}$. Otherwise, we set d_{m+1} be the least cardinal that belongs to $C^{(n)}$ above d_m. We claim that $\kappa = \sup\{d_m : m \in \omega\} > \gamma$ belongs to $C^{(n+1)}$.

First, note that, as κ is a limit of cardinals that belong to $C^{(n)}$, by the induction hypothesis, κ also belongs to $C^{(n)}$. Moreover, let $b \in V_\kappa$ and, for some $m \in \omega$, let $\phi_m(x)$ be a Σ_{n+1} formula, which of course is listed in the enumeration above. To check that κ does indeed belong to $C^{(n+1)}$, we have to show that $\phi_m(b)$ holds if and only if $V_\kappa \vDash \phi_m(b)$. Now, it is easy to see that if $V_\kappa \vDash \phi_m(b)$, then, by upwards absoluteness, $\phi_m(b)$ holds. For the other direction, if $\phi_m(b)$ holds, then, there exists an a (somewhere in V) such that $\psi_m(a, b)$ holds. But, from the construction of κ, we have included in V_κ at least one such witness a' for ψ_m and, since $\kappa \in C^{(n)}$, we have that $V_\kappa \vDash \psi_m(a', b)$. Hence, $V_\kappa \vDash \phi_m(b)$ and so $\kappa \in C^{(n+1)}$.

Lastly, it is easy to see that $C^{(n+1)}$ is also closed, completing that way the induction. □

Concerning the complexity of definability of these notions, we have that "$\alpha \in C^{(0)}$" is Δ_0 definable and, in general, for every $n \geq 1$, "$\alpha \in C^{(n)}$" is Π_n-definable[1] since $\alpha \in C^{(n)}$ if and only if the following holds:

$$\alpha \in C^{(n-1)} \wedge \forall \phi(x) \in \Sigma_n \forall b \in V_\alpha (\models_n \phi(b) \rightarrow V_\alpha \models \phi(b))$$

This is optimal, since, if for $n > 0$ we had that $C^{(n)}$ is Σ_n, then, if α was the least ordinal in $C^{(n)}$, the statement "$\exists x(x \in C^{(n)})$" would hold in V_α, which of course is a contradiction.

Another thing is that, if $\mathcal{C}$ is a Σ_n club class of ordinals, then it contains $C^{(n)}$. For, suppose that $\mathcal{C}$ is a Σ_n club class of ordinals and $\alpha \in C^{(n)}$. Then, $\mathcal{C}$ is unbounded below α since, for every $\beta < \alpha$, the sentence

$$\exists \gamma(\beta < \gamma \wedge \gamma \in \mathcal{C})$$

[1] Recall that, for some ordinal α, the (abbreviated) statement "$x = V_\alpha$" is Π_1.

is Σ_n in the parameter β and is true in V. Hence, it is also true in V_α. Now, by the closure of $\mathcal{C}$, we have that $\alpha \in \mathcal{C}$. Similarly, every club proper class of ordinals that is $\boldsymbol{\Sigma_n}$ contains all $\alpha \in C^{(n)}$ that are greater than the rank of the parameters in some $\boldsymbol{\Sigma_n}$-definition of $\mathcal{C}$.

Finally, note that for $n \geq 0$, we have that $C^{(n+1)} \subseteq C^{(n)}$. Moreover, the least ordinal, α, that belongs in $C^{(n)}$ does not belong to $C^{(n+1)}$ since, otherwise, the sentence $\exists x(x \in C^{(n)})$ would be Σ_{n+1} and so, it would hold in V_α; a contradiction. Thus, for $n \geq 0$, we have that $C^{(n+1)} \subsetneq C^{(n)}$.

Having presented the ordinal classes $C^{(n)}$ and their basic properties, we are now ready to introduce one of the simplest examples of $C^{(n)}$-cardinals; that of $C^{(n)}$-measurables.

Definition 4.3. For $n \geq 0$, a cardinal κ is $C^{(n)}$*-measurable* if there is an elementary embedding $j : V \prec M$ with M transitive, $cp(j) = \kappa$ and $j(\kappa) \in C^{(n)}$.

Note that, if U is a κ-complete ultrafilter over κ and $j_U : V \prec M \cong Ult(V, U)$ is the ultrapower embedding obtained from U, then by Proposition 2.9, $j(\kappa)$ is not even a cardinal. However, exploiting the iterated ultrapower construction, for $\alpha > 2^\kappa$, if we take the α-th iterated ultrapower embedding $j_\alpha : V \prec M_a$, we have that $j_\alpha(\kappa) = \alpha$ (cf. [10, Lem. 19.15]). Hence, since for every $n \geq 0$, $C^{(n)}$ is a proper class, we can always find an elementary embedding for which the image of the critical point belongs to $C^{(n)}$. In other words, we have that if κ is measurable, then, it is $C^{(n)}$-measurable for every $n \geq 0$. This means that, in the case of measurable cardinals, the additional requirement that $j(\kappa) \in C^{(n)}$ does not yield a stronger large cardinal notion and so, it is not of any particular interest. However, as we will shortly see, this changes for larger large cardinals.

4.2 $C^{(n)}$-superstrongs

Climbing a bit higher in the large cardinal hierarchy, we now explore the $C^{(n)}$ counterpart of superstrongness.

Definition 4.4. For $n \geq 0$, a cardinal κ is $C^{(n)}$*-superstrong* if there exists an elementary embedding $j : V \prec M$ with M transitive, $cp(j) = \kappa$, $V_{j(\kappa)} \subseteq M$ and $j(\kappa) \in C^{(n)}$.

Observe that, for $n \geq 0$, every $C^{(n)}$-superstrong cardinal belongs to $C^{(n)}$: for, suppose that κ is a $C^{(n)}$-superstrong cardinal and $j : V \prec M$ is a witnessing elementary embedding. Then, recalling that $j \restriction V_\kappa$ is the identity, it is easy to see that $V_\kappa \prec V_{j(\kappa)}$ and thus, since $j(\kappa) \in C^{(n)}$, we have that $\kappa \in C^{(n)}$.

Proposition 4.5. Suppose κ is a superstrong cardinal and $j : V \prec M$ a witnessing elementary embedding. Then, $j(\kappa) \in C^{(1)}$, i.e., every superstrong cardinal is $C^{(1)}$-superstrong.

Proof. Obviously $\kappa \in C^{(1)}$ and so, by elementarity, we have that $M \models j(\kappa) \in C^{(1)}$. Moreover, since $V^M_{j(\kappa)} = V_{j(\kappa)}$, we have that $j(\kappa)$ does indeed belong to $C^{(1)}$. □

As for the defining complexity of $C^{(n)}$-superstrongness, recalling Proposition 3.2, we have that for $n \geq 0$, κ is $C^{(n)}$-superstrong if and only if

$$\exists\lambda\exists\mu\exists E\big(\kappa < \lambda < \mu \wedge \mu \in C^{(n)} \wedge \text{“}E \text{ is a } (\kappa,\lambda)\text{-extender”} \wedge E \in V_\mu \wedge \\ V_\mu \models (j_E(\kappa) \in C^{(n)} \wedge V_{j_E(\kappa)} \subseteq M_E)\big)$$

Hence, recalling that “$x \in C^{(n)}$” is Π_n and the fact that checking if E is indeed an extender is something that can be verified locally, we have that the property of being $C^{(n)}$-superstrong is Σ_{n+1}.

Now, contrary to the case of $C^{(n)}$-measurables, the following proposition implies that the first $C^{(n+1)}$-superstrong cardinal, if it exists, is not $C^{(n)}$-superstrong; hinting that way that $C^{(n)}$-superstrong cardinals form a hierarchy of increasing consistency strength.

Proposition 4.6. For $n \geq 1$, if κ is $C^{(n+1)}$-superstrong, then, there is a normal ultrafilter U over κ such that

$$\{\alpha < \kappa : \text{“}\alpha \text{ is } C^{(n)}\text{-superstrong”}\} \in U$$

Proof. Let κ be a $C^{(n+1)}$-superstrong cardinal and $j : V \prec M$ a witnessing elementary embedding. Then, since $j(\kappa) \in C^{(n+1)}$, we have that $V_{j(\kappa)} \models$ “κ is $C^{(n)}$-superstrong” and, moreover, since $\kappa \in C^{(n+1)}$, by elementarity it follows that $M \models j(\kappa) \in C^{(n+1)}$. Hence, since $V^M_{j(\kappa)} = V_{j(\kappa)}$, we have that $M \models$ “κ is $C^{(n)}$-superstrong”. Lastly, let U be the normal ultrafilter constructed through j. Then, by a standard reflection argument on U, the set

$$\{\alpha < \kappa : \text{“}\alpha \text{ is } C^{(n)}\text{-superstrong”}\}$$

belongs in U. □

One may now wonder where do $C^{(n)}$-superstrong cardinals lay in terms of the usual large cardinal hiearchy. The following definition will help us answer this question.

Definition 4.7. For $n \geq 0$, a cardinal κ is called λ-$C^{(n)}$*-extendible*, for some $\lambda > \kappa$, if there exists a $\mu > \lambda$ and an elementary embedding $j : V_\lambda \prec V_\mu$ with $cp(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. Moreover, we say that κ is $C^{(n)}$*-extendible* if it is λ-$C^{(n)}$-extendible for all $\lambda > \kappa$.

We now give an upper bound of the consistency strength of $C^{(n)}$-superstrong cardinals.

Theorem 4.8. For $n \geq 1$, if κ is 2^κ-supercompact and belongs to $C^{(n)}$, then, there is normal ultrafilter U over κ such that

$$\{\alpha < \kappa : \text{“}\alpha \text{ is } C^{(n)}\text{-superstrong”}\} \in U.$$

Proof. Let W be a normal and fine ultrafilter on $\mathcal{P}_\kappa(2^\kappa)$ and $j_W : V \prec M$ the elementary embedding constructed from W. Observe that, $j_W \restriction V_{\kappa+1} : V_{\kappa+1} \to (V_{j_W(\kappa)+1})^M$ is an elementary embedding and, by the closure of M under 2^κ-sequences, we have that

$j_W \restriction V_{\kappa+1} \in M$. Moreover, M (correctly) thinks that $j \restriction V_{\kappa+1}$ is an elementary embedding and, in addition, by elementarity M also thinks that $j(\kappa) \in C^{(n)}$. Thus, $M \vDash$ "κ is $\kappa+1$-$C^{(n)}$-extendible", and if U is the ultrafilter derived from j_W, then, by a standard reflection argument, we get that

$$\{\alpha < \kappa : \text{“}\alpha \text{ is } \alpha+1\text{-}C^{(n)}\text{-extendible”}\} \in U.$$

Lastly, using similar arguments as in the proof of Proposition 3.3, one can show that if a cardinal α is $\alpha+1$-$C^{(n)}$-extendible, then α is $C^{(n)}$-superstrong. □

At this point, let us briefly mention that the $C^{(n)}$ counterparts of various (usual) large cardinals have also been explored (e.g. $C^{(n)}$-strong, $C^{(n)}$-Woodins, $C^{(n)}$-huge, etc.). However, to present them here would be a distraction from our main purpose. As a final example, which will be of great use to us, we will now present $C^{(n)}$-extendible cardinals; a pivotal concept in the study of $C^{(n)}$-cardinals.

4.3 $C^{(n)}$-extendibles

We have already introduced the notion of $C^{(n)}$-extendibility (Definition 4.7) and so, we begin with some straightforward facts about $C^{(n)}$-extendible cardinals.

Proposition 4.9. Every extendible cardinal is $C^{(1)}$-extendible.

Proof. Suppose κ is an extendible cardinal and λ is a cardinal greater than κ. Let λ' be a $C^{(1)}$ cardinal with $\lambda' > \lambda$ and $j : V_{\lambda'} \prec V_\mu$ a witnessing elementary embedding of the λ'-extendibility of κ. Obviously, since $\lambda' \in C^{(1)}$, we have that $\mu \in C^{(1)}$. Furthermore, since κ is inaccessible, we have that $\kappa \in C^{(1)}$. So, remembering that the statement "$\kappa \in C^{(1)}$" is Π_1, we have that $V_{\lambda'} \vDash \kappa \in C^{(1)}$. By elementarity, we get that $V_\mu \vDash j(\kappa) \in C^{(1)}$ and, since $\mu \in C^{(1)}$, $j(\kappa)$ is indeed a $C^{(1)}$ cardinal. □

Observe that, if κ is $C^{(n)}$-extendible, for some $n \geq 0$, and j is an elementary embedding witnessing the λ-extendibility of κ for some $\lambda > \kappa$, then, since $V_\kappa \prec V_{j(\kappa)}$ we have that $\kappa \in C^{(n)}$. In fact, κ possesses even stronger reflection properties.

Theorem 4.10. For $n \geq 1$, if κ is $C^{(n)}$-extendible, then $\kappa \in C^{(n+2)}$.

Proof. We prove this by induction (in the meta-theory). Once again, recall that extendible cardinals belong to $C^{(3)}$ and so, we are covered for the base case. Now, for $n > 1$, suppose that κ is $C^{(n)}$-extendible, $\phi(x)$ is a Σ_{n+2} formula of the form $\exists y \psi(x, y)$, where $\psi(x, y)$ is Π_{n+1}, and a is a parameter that belongs in V_κ.

If $V_\kappa \vDash \phi(a)$, then since by the induction hypothesis $\kappa \in C^{(n+1)}$, by upwards absoluteness $\phi(a)$ holds. On the other direction, if $\phi(a)$ holds, then there is a b (somewhere in V) such that $\psi(a, b)$ holds. Pick $\lambda > \kappa$ such that $b \in V_\lambda$ and, for some μ, let $j : V_\lambda \prec V_\mu$ be an elementary embedding witnessing the λ-$C^{(n)}$-extendibility of κ. Then, since $j(\kappa)$ is a cardinal greater than λ that belongs in $C^{(n)}$ and $\psi(a,b)$ is a Π_{n+1} formula that holds in V, by downwards absoluteness, we get that $V_{j(\kappa)} \models \psi(a,b)$. In other words, $V_{j(\kappa)} \models \phi(a)$ and, by elementarity, it follows that $V_\kappa \models \phi(a)$. □

As for the defining complexity of $C^{(n)}$-extendibility, note that for $n \geq 0$, a cardinal κ and a $\lambda > \kappa$, κ is λ-$C^{(n)}$-extendible if and only if

$$\exists \mu \exists j \big(\text{“}j : V_\lambda \to V_\mu \text{ is elementary ”} \wedge cp(j) = \kappa \wedge j(\kappa) > \lambda \wedge j(\kappa) \in C^{(n)}\big).$$

So, “κ is λ-$C^{(n)}$-extendible” is Σ_{n+1} expressible and thus, “κ is $C^{(n)}$-extendible” is Π_{n+2}. With that being said, we now proceed with a few further properties of those cardinals.

The following proposition (the first claim) suggests that $C^{(n)}$-extendible cardinals also form a hierarchy.

Proposition 4.11. For $n \geq 1$, the following hold:

1. If κ is $C^{(n)}$-extendible and $\kappa+1$-$C^{(n+1)}$-extendible, then the set of $C^{(n)}$-extendible cardinals is unbounded below κ. Hence, the first $C^{(n)}$-extendible cardinal, if it exists, is not $\kappa+1$-$C^{(n+1)}$-extendible.
2. If there exists a $C^{(n+2)}$-extendible cardinal, then there exists a proper class of $C^{(n)}$-extendible cardinals.
3. The existence of a $C^{(n+1)}$-extendible cardinal κ does not imply the existence of a $C^{(n)}$-extendible greater than κ.

Proof. For 1, let κ be a cardinal as above and $j : V_{\kappa+1} \prec V_{j(\kappa)+1}$ be an elementary embedding that witnesses the $\kappa+1$-$C^{(n+1)}$-extendibility of κ. By downwards absoluteness, $V_{j(\kappa)} \models$ “κ is $C^{(n)}$-extendible”. Thus, for every $\alpha < \kappa$, we have that $V_{j(\kappa)} \models \exists \beta > \alpha$(“$\beta$ is $C^{(n)}$-extendible”) and so, by elementarity, for every fixed $\alpha < \kappa$, there is a $\beta > \alpha$ such that

$$V_\kappa \models \beta > \alpha \wedge \text{“}\beta \text{ is } C^{(n)}\text{-extendible”}.$$

Lastly, by Theorem 4.10, such a β is indeed a $C^{(n)}$-extendible cardinal.

For 2, note that from 1, we have that if κ is a $C^{(n+2)}$-extendible cardinal, then the set of $C^{(n)}$-extendible cardinals below κ is unbounded. Moreover, from Theorem 4.10, $\kappa \in C^{(n+4)}$ and, since being $C^{(n)}$-extendible is a Π_{n+2} property, those cardinals below κ are indeed $C^{(n)}$-extendible cardinals.

For 3, let κ be a $C^{(n+1)}$-extendible cardinal. There are two cases: either there are no $C^{(n)}$-extendible cardinals above κ, or there is at least one. If it is the former case, we are done. If it is the latter, let λ be the least $C^{(n)}$-extendible cardinal above κ. Then, we have that

$$V_\lambda \models ZFC + \text{“}\kappa \text{ is } C^{(n+1)}\text{-extendible”} + \neg \exists \lambda' (\lambda' > \kappa \wedge \text{“}\lambda' \text{ is } C^{(n)}\text{-extendible”}),$$

since λ is an inaccessible cardinal that, by Theorem 4.10, belongs in $C^{(n+2)}$ and the properties of being $C^{(n+1)}$-extendible and $C^{(n)}$-extendible are Π_{n+3} and Π_{n+2} respectively. In other words, V_λ, a model of ZFC where there is a $C^{(n+1)}$-extendible cardinal, does not satisfy the existence of a $C^{(n)}$-extendible cardinal above κ. Thus, in both cases, even though we can not decide in ZFC which is the case, we get the required result. $\square$

Remark. The statement 3 in the preceding proposition should be understood as the following relative consistency statement:

$$Con\big(ZFC + \exists\kappa(\text{“}\kappa \text{ is } C^{(n+1)}\text{-extendible”})\big)$$
$$\Downarrow$$
$$Con\big(ZFC + \exists\kappa(\text{“}\kappa \text{ is } C^{(n+1)}\text{-extendible”}) + \neg\exists\lambda(\lambda > \kappa \wedge \text{“}\lambda \text{ is } C^{(n)}\text{-extendible”})\big)$$

Before continuing further with the case of $C^{(n)}$-extendible cardinals, let us open a (small) parenthesis. One more $C^{(n)}$-cardinal notion is that of $C^{(n)}$-supercompactness: a cardinal κ is *λ-$C^{(n)}$-supercompact*, for some $n \geq 0$ and some $\lambda > \kappa$, if there is an elementary embedding $j : V \prec M$, with M transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ and $j(\kappa) \in C^{(n)}$. Moreover, κ is *$C^{(n)}$-supercompact* if and only if it is λ-$C^{(n)}$-supercompact for every $\lambda > \kappa$.

The notion of $C^{(n)}$-supercompactness does not seem to have interesting reflection properties so far and so, to avoid unnecessary burden (for the, possible, readers, as well as the, definite, writer), we skip any further investigation. However, as we will shortly see, $C^{(n)}$-supercompactness together with the notion of superstrongness, will provide another useful characterization of $C^{(n)}$-extendibility. Lastly, let us just mention that, by utilizing the theory of (Martin-Steel) extenders, the property of being λ-$C^{(n)}$-supercompact is Σ_{n+1} and thus, being (fully) $C^{(n)}$-supercompact is a Π_{n+2} property.

We now “close the parenthesis” and, as the following definition suggests, we join the notions of ($C^{(n)}$-) supercompactness and superstrongness, enhancing that way the reflection properties of the target model. Let us also mention that, until the end of this chapter, the content we will now present is, unless otherwise stated, due to Tsaprounis [15].

Definition 4.12. A cardinal κ is called *jointly λ-supercompact and θ-superstrong*, for some $\lambda, \theta \geq \kappa$, if there is an elementary embedding $j : V \prec M$ with M transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ and $V_{j(\theta)} \subseteq M$.

If we do not specify the parameter(s) λ or (and) θ, then, the corresponding large cardinal notions are implied. For example, a cardinal κ is *jointly supercompact and θ-superstrong*, for some $\theta \geq \kappa$, if it is jointly λ-supercompact and θ-superstrong for every $\lambda \geq \kappa$. Moreover, note that κ-superstrongness is the usual superstrongness. Needless to say, we also have the $C^{(n)}$ versions of this notion.

Now, observe that, if κ is the least supercompact cardinal, then, it can not be the case that κ is also jointly λ-supercompact and κ-superstrong, for any λ. For, suppose that κ, the least supercompact cardinal, is also jointly λ-supercompact and κ-superstrong, and $j : V \prec M$ is a witnessing elementary embedding. Then, from Proposition 4.5, we have that $j(\kappa) \in C^{(1)}$ and since being supercompact is Π_2, by downwards absoluteness, we get that $V_{j(\kappa)} \models$ "κ is supercompact". Noting that $V_\kappa \prec V_{j(\kappa)}$, it follows that there is some $\mu < \kappa$ such that $V_\kappa \models$ "μ is supercompact". Lastly, by Theorem 3.6, $\kappa \in C^{(2)}$ and hence, μ is indeed a supercompact cardinal; a contradiction to the minimality of κ.

Joint $C^{(n)}$-supercompactness and superstrongness has strong reflection properties. More precisely, the following holds.

Proposition 4.13. For $n \geq 0$, if κ is jointly $C^{(n)}$-supercompact and κ-superstrong, then $\kappa \in C^{(n+2)}$.

Proof. Similar to the proof of Theorem 4.10. □

Just as in the case of "simple" supercompactness (recall Proposition 3.7), we have the following corollary.

Corollary 4.14. For $n \geq 0$, if κ is jointly $C^{(n)}$-supercompact and κ-superstrong, and $\alpha < \kappa$ is $< \kappa$-$C^{(n)}$-supercompact, then α is $C^{(n)}$-supercompact.

Proof. Recalling that $C^{(n)}$-supercompact is a Π_{n+2} property, and using the previous proposition, the proof follows easily. □

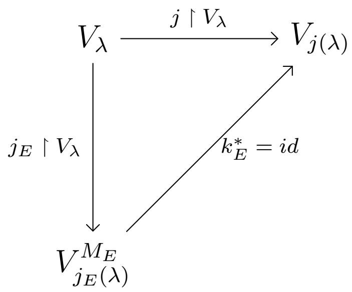
We now want to investigate the relation between $C^{(n)}$-supercompact and $C^{(n)}$-extendible cardinals. For this, the concept of joint supercompactness and superstrongness will play a crucial role.

Theorem 4.15. For $n \geq 0$, suppose that κ is $\lambda+1$-$C^{(n)}$-extendible for some $\lambda > \kappa$ with $\beth_\lambda = \lambda$ and $cof(\lambda) > \kappa$. Then, κ is jointly λ-$C^{(n)}$-supercompact and λ-superstrong.

Proof. Fix $n \geq 0$ and a $\lambda > \kappa$ such that $\beth_\lambda = \lambda$ and $cof(\lambda) > \kappa$. Moreover, let $j : V_{\lambda+1} \prec V_{j(\lambda)+1}$ be an elementary embedding that witnesses the $\lambda+1$-$C^{(n)}$-extendibility of κ.

Now, consider the $(\kappa, j(\lambda))$-extender E derived from j, which is of the form $\langle E_a : a \in [j(\lambda)]^{<\omega}\rangle$, where for every $a \in [j(\lambda)]^{<\omega}$, E_a is a κ-complete ultrafilter over $[\lambda]^{|a|}$. First, observe that the definition of E is a valid definition and that the situation here is similar to that of the proof of Proposition 3.3. More precisely, we have an elementary embedding between sets and so, we have to check that E is indeed a $(\kappa, j(\lambda))$-extender. This is done in a similar manner to the proof of Proposition 3.3 and so we choose to skip those details. The conclusion is that E is indeed a $(\kappa, j(\lambda))$-extender and this is verified inside $V_{j(\lambda)+1}$.

So, let $j_E : V \prec M_E$ be the extender elementary embedding with $cp(j_E) = \kappa$. Once again, along the same lines of Proposition 3.3, we define an $\{\in\}$-embedding, which in fact is the identity function, $k_E^* : V_{j_E(\lambda)}^{M_E} \to V_{j(\lambda)}$, by letting $k_E^*([a,[f]]) = j(f)(a)$ for all $[a,[f]] \in V_{j_E(\lambda)}^{M_E}$, where $a \in [j(\lambda)]^{<\omega}$ and $f : [\lambda]^{|a|} \to V_\lambda$. Hence, as before, we have the following commutative diagram.



It now follows that $V_{j(\lambda)} = V^{M_E}_{j_E(\lambda)} \subseteq M_E$, which, by Proposition 3.2, implies that κ is superstrong. Furthermore, since k^*_E is the identity, we have that for every $\alpha \leq \lambda$, $j_E(\alpha) = j(\alpha)$ and, noting that $cof(\lambda) > \kappa$ implies that $cof(j(\lambda)) > \lambda$, we have that $j_E"\lambda = j"\lambda \in V_{j(\lambda)}$. Consequently, we have that $j_E"\lambda \in M_E$. Using this fact, we will show that M_E is closed under λ-sequences, concluding that way the proof.

Recall that

$$M_E = \{j_E(f)(a) : a \in [j(\lambda)]^{<\omega}, f : [\lambda]^{|a|} \to V, f \in V\}.$$[2]

With that in mind, let $\langle j_E(f_\xi)(a_\xi) : \xi < \lambda\rangle$ be a λ-sequence of elements of M_E; we will show that it belongs in M_E. Now, since $cof(j(\lambda)) > \lambda$, we have that $\langle a_\xi : \xi < \lambda\rangle \in V_{j(\lambda)} \subseteq M_E$. Hence, it suffices to show that $\langle j_E(f_\xi) : \xi < \lambda\rangle \in M_E$ since, if this is the case, M_E can compute the desired sequence by evaluating pointwise the functions $j_E(f_\xi)$ at the corresponding a_ξ. Now, observe that since $j_E"\lambda \in M_E$, we also have that $j_E \restriction \lambda \in M_E$, where $j_E \restriction \lambda : \lambda \to j"\lambda$. Furthermore, $\langle f_\xi : \xi < \lambda\rangle$ is a function (written in sequence form) from λ to V and, by elementarity, $G = j_E(\langle f_\xi : \xi < \lambda\rangle)$ is a function from $j_E(\lambda)$ to M_E, which obviously belongs to M_E. Lastly, we define in M_E the function $F = G \circ (j_E \restriction \lambda) : \lambda \to M_E$, which belongs to M_E, and observe that $F = \langle j_E(f_\xi) : \xi < \lambda\rangle$ since, for every $\xi < \lambda$, we have that

$$F(\xi) = G(j_E(\xi)) = j_E(\langle f_\xi : \xi < \lambda\rangle)(j_E(\xi)) = j_E(\langle f_\xi : \xi < \lambda\rangle(\xi)) = j_E(f_\xi).$$

□

We immediately get the following corollary.

Corollary 4.16. For $n \geq 0$ and κ a cardinal, if κ is $C^{(n)}$-extendible, then κ is jointly $C^{(n)}$-supercompact and superstrong.

Observe that, from the previous corollary it follows that $C^{(n)}$-extendibility implies $C^{(n)}$-supercompactness. In particular, as we mentioned in the section of supercompactness, for $n = 1$, we get that every extendible cardinal is supercompact.

In the opposite direction, we also have that joint $C^{(n)}$-supercompactness and superstrongness implies $C^{(n)}$-extendibility.

[2]Cf. [17, Prop. 3.4]

Theorem 4.17. For $n \geq 0$, if κ is jointly $C^{(n)}$-supercompact and κ-superstrong, then κ is $C^{(n)}$-extendible.

Proof. We will first deal with the general case for $n \geq 1$. So, fix some $n \geq 1$ and suppose that κ is jointly $C^{(n)}$-supercompact and κ-superstrong. Moreover, fix some $\lambda > \kappa$ with $\lambda \in C^{(n+2)}$ and let $j : V \prec M$ be an elementary embedding witnessing the fact that κ is jointly λ-$C^{(n)}$-supercompact and κ-superstrong.

By elementarity, we have that $M \vDash j(\lambda) \in C^{(n+2)}$ and since, by Proposition 4.13, κ belongs in $C^{(n+2)}$, we also have that $M \vDash j(\kappa) \in C^{(n+2)}$. By the closure of M, and since $|V_\lambda| = \lambda$, we have that $j \restriction V_\lambda : V_\lambda \to V^M_{j(\lambda)}$ belongs in M and thus, M witnesses the $< \lambda$-$C^{(n)}$-extendibility of κ, that is,

$$M \vDash \text{“}\kappa \text{ is } < \lambda\text{-}C^{(n)}\text{-extendible”}.$$

Furthermore, since $j(\kappa) \in C^{(n)}$, by downwards absoluteness we get that $V_{j(\kappa)} \vDash \lambda \in C^{(n+1)}$. Additionally, from superstrongness we have that $V_{j(\kappa)} \subseteq M$ and since $M \vDash j(\kappa) \in C^{(n+2)}$, we have that $M \vDash \lambda \in C^{(n+1)}$. So, recalling that the defining complexity of the property of being $< \lambda$-$C^{(n)}$-extendible is Σ_{n+1}, we have that

$$M \vDash \big(V_\lambda \vDash \text{“}\kappa \text{ is } C^{(n)}\text{-extendible”}\big).$$

Therefore, since $V_\lambda \subseteq M$, we get that $V_\lambda \vDash$ “κ is $C^{(n)}$-extendible”, which in turn, remembering that $\lambda \in C^{(n+2)}$, yields that κ is indeed a $C^{(n)}$-extendible cardinal.

Now, for $n = 0$, we want to show that if κ is jointly supercompact and κ-superstrong, then it is also an extendible cardinal. This time, we have to pick a $\lambda > \kappa$ that belongs in $C^{(3)}$, since the property of being extendible is Π_3 and λ should be correct enough in order for κ to “truly” be an extendible cardinal. Moreover, recall that, by Theorem 3.6, if κ is supercompact, then $\kappa \in C^{(2)}$ which, using similar arguments as above, yields that $M \vDash \lambda \in C^{(2)}$. With that being said, the proof is almost identical to that above and so, we leave the details for the reader. □

By grouping the results of Theorems 4.16 and 4.17, we get another characterization of $C^{(n)}$-extendibility:

Corollary 4.18. For $n \geq 0$, a cardinal κ is $C^{(n)}$-extendible if and only if it is jointly $C^{(n)}$-supercompact and κ-superstrong.

We will now finish this chapter with one last theorem, which suggests that the consistency of all the $C^{(n)}$-cardinal notions we have considered is implied by the existence of an almost huge cardinal.[3] Firstly though, consider the following notion, closely related to that of $C^{(n)}$-extendibles.

[3]A cardinal κ is almost huge if and only if there is an elementary embedding $j : V \prec M$ with $cp(j) = \kappa$ and ${}^{\gamma}M \subseteq M$, for every $\gamma < j(\kappa)$.

Definition 4.19 ([2]). For $n \geq 0$ and $\lambda \in C^{(n)}$, a cardinal κ is λ-$C^{(n)+}$-*extendible* if there exists $\mu \in C^{(n)}$ and an elementary embedding $j : V_\lambda \prec V_\mu$ with $cp(j) = \kappa, j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. As someone would expect, κ is $C^{(n)+}$-*extendible* if it is λ-$C^{(n)+}$-extendible for every $\lambda \in C^{(n)}$ with $\lambda > \kappa$.

Theorem 4.20. Suppose that κ is an almost huge cardinal and $j : V \prec M$ is a witness of that fact. Moreover, let U be the normal ultrafilter derived from j. Then, for every $n \in \omega$, we have that:

$$V_{j(\kappa)} \vDash \text{“}\kappa \text{ is } C^{(n)+}\text{-extendible”} \wedge \{\alpha < \kappa : \text{“}\alpha \text{ is } C^{(n)+}\text{-extendible”}\} \in U$$

Proof. First, we claim that $j(\kappa)$ is inaccessible and that $V_{j(\kappa)} \subseteq M$. To see this, we prove by induction that for every $\alpha < j(\kappa)$, the following conjuction holds:

$$|V_\alpha| < j(\kappa) \wedge V_\alpha \subseteq M.$$

Note that, if this is true, then we are done with the proof of the claim. So, proceeding by induction, let $\alpha < j(\kappa)$ and suppose that $|V_\alpha| < j(\kappa)$ and that $V_\alpha \subseteq M$; we will show that the same is true for $\alpha + 1$. Now, since $M \vDash$ “$j(\kappa)$ is inaccessible”, we get that

$$|V_{\alpha+1}| = 2^{|V_\alpha|} \leq (2^{|V_\alpha|})^M < j(\kappa),$$

which gives the first part of the conjuction. As for the second part, let $x \subseteq V_\alpha$. Now, every $y \in x$ belongs to V_α and, by the induction hypothesis, to M. Moreover, we have that $|x| \leq |V_\alpha| < j(\kappa)$ and thus, by the closure of M under sequences of size less than $j(\kappa)$, it follows that $x \in M$. In other words, $V_{\alpha+1} \subseteq M$. Finally, one treats the limit case similarly, since again by the closure of M, it is easy to see that $j(\kappa)$ is a regular cardinal (in V), that way completing the induction.

Now, proceeding with the rest of the proof, fix some $n \in \omega$ and let λ be a cardinal in between κ and $j(\kappa)$ such that $V_{j(\kappa)} \vDash \lambda \in C^{(n)}$. Note that, such a λ exists since $V_{j(\kappa)}$ is a model of ZFC and thus, in the eyes of $V_{j(\kappa)}$, $C^{(n)}$ cardinals form a proper class. Now, by the closure of M under λ-sequences and the inaccessibility of $j(\kappa)$, we have that $j \restriction V_\lambda \in M$. Moreover, M thinks that $j \restriction V_\lambda : V_\lambda \to V_{j(\lambda)}$ is an elementary embedding with $cp(j \restriction V_\lambda) = \kappa$, $(j \restriction V_\lambda)(\kappa) > \lambda$ and $j(\lambda) < j(j(\kappa))$. Lastly, by elementarity, M also satisfies that $V_{j(j(\kappa))} \vDash j(\lambda) \in C^{(n)}$.

Let $\phi(\lambda, \mu, \kappa)$ be a formula that asserts that “there exists a λ-extendibility embedding e for κ with $\mu = e(\lambda)$”. By the previous paragraph, for every λ in between κ and $j(\kappa)$ with $V_{j(\kappa)} \vDash \lambda \in C^{(n)}$, we have that

$$M \vDash \exists \mu < j(j(\kappa))\big(\phi(\lambda, \mu, \kappa) \wedge V_{j(j(\kappa))} \vDash \mu \in C^{(n)}\big).$$

So, by the usual reflection argument of the normal ultrafilter, we have that the set, A, of ordinals $\alpha < \kappa$ such that, for all λ in between α and κ, it holds that

$$\big(V_\kappa \vDash \lambda \in C^{(n)}\big) \to \exists \mu < j(\kappa)\big(\phi(\lambda, \mu, \alpha) \wedge V_{j(\kappa)} \vDash \mu \in C^{(n)}\big)$$

belongs to U. Fix any such $\alpha \in A$, a λ in between α and κ with $V_\kappa \models \lambda \in C^{(n)}$ and an $\mu < j(\kappa)$ witnessing that $\alpha \in A$.

Since $\mu < j(\kappa)$, once again by the inaccessibility of $j(\kappa)$, the elementary embedding that witnesses the λ-extendibility of α belongs to $V_{j(\kappa)}$ and thus, we have that

$$V_{j(\kappa)} \models \exists\mu\big(\phi(\lambda, \mu, \alpha) \wedge \mu \in C^{(n)}\big).$$

Hence, by elementarity, for any such $\alpha \in A$ and any fixed λ in between α and κ with $V_\kappa \models \lambda \in C^{(n)}$, we have that there exists an $\mu < \kappa$ such that

$$V_\kappa \models \phi(\lambda, \mu, \alpha) \wedge \mu \in C^{(n)}.$$

It now follows that A is a subset of the set B of all ordinals $\alpha < \kappa$ such that $V_\kappa \models \forall\lambda > \alpha\big(\lambda \in C^{(n)} \rightarrow \exists\mu(\phi(\lambda, \mu, \alpha) \wedge \mu \in C^{(n)})\big)$. So, B also belongs in U and, again by the usual reflection argument of U (this time in the opposite direction), we get that

$$V_{j(\kappa)} \models \forall\lambda > \kappa\big(\lambda \in C^{(n)} \rightarrow \exists\mu(\phi(\lambda, \mu, \kappa) \wedge \mu \in C^{(n)})\big).$$

Finally, for any λ in between κ and $j(\kappa)$ with $V_{j(\kappa)} \models \lambda \in C^{(n)}$, if we take any witnessing extendibility embedding $e : V_\lambda \prec V_\mu$ for κ in $V_{j(\kappa)}$, we have that $V_{j(\kappa)} \models \kappa, \lambda, \mu \in C^{(n)}$ and thus, $e(\kappa) \in C^{(n)}$.[4] In other words, we have that $V_{j(\kappa)} \models$ "κ is $C^{(n)+}$-extendible" and so the set $\{\alpha < \kappa : V_\kappa \models$ "α is $C^{(n)+}$-extendible"$\}$ belongs in U. Lastly, from Proposition 4.10, we have that $V_{j(\kappa)} \models \kappa \in C^{(n+2)}$ and thus,

$$V_{j(\kappa)} \models \{\alpha < \kappa : \text{“}\alpha \text{ is } C^{(n)+}\text{-extendible”}\} \in U.$$

□

Remark. As a matter of fact, the notions of $C^{(n)}$-extendible and $C^{(n)+}$-extendible cardinals are not just "closely related", but are in fact equivalent. Even though this equivalence was believed to be true from the beginning of the exploration of $C^{(n)}$-cardinals, no proof of it had been found. A first attempt was made by Bagaria and Brooke-Taylor in [4] and, some years later, it was finally proved by Tsaprounis in [16] and, independently and in the context of a different study, by Gitman and Hamkins in [8].

Next, we move forward to our main goal.

[4]In general, for a $j : V_\lambda \prec V_\mu$ with $cp(j) = \kappa$, if $\kappa, \lambda, \mu \in C^{(n)}$, then it holds that $j(\kappa) \in C^{(n)}$, since we have that

$$\kappa \in C^{(n)} \Leftrightarrow V_\lambda \models \kappa \in C^{(n)} \Leftrightarrow V_\mu \models j(\kappa) \in C^{(n)} \Leftrightarrow j(\kappa) \in C^{(n)}.$$

5. STRUCTURAL REFLECTION

In this, final, chapter we explore various reflection phenomena that take place in the area right beneath Vopěnka's Principle. In particular, we present a level-by-level correspondence between strata of Vopěnka's Principle and the hierarchy of $C^{(n)}$-extendible cardinals.

Once again, the content of this chapter is due to Bagaria [2].

5.1 On Vopěnka's Principle

Vopěnka's Principle was first introduced in the 1960's by the Czech mathematician Petr Vopěnka. Initially, Vopěnka presented his axiom as a kind of a joke; an act to tease set-theorists for the plethora of large cardinal notions they were introducing at that time. He expected that such a strong axiom can only lead to a contradiction and, in fact, he himself tried to prove that this was the case. But, contrary to his belief, not only such a proof has not (yet) been found, but it has also become clear that there is a deep connection between Vopěnka's hypothesis and various areas of mathematics; category theory being an archetype (cf. [1, ch. 6.], [5]). As for the context of large cardinals, a well known result is that ZFCplus Vopěnka's Principle implies the existence of a stationary proper class of extendible cardinals [14]. All this led to Vopěnka's axiom being placed in its rightful position in the large cardinal hierarchy (Figure 1.1). From then, the terminology Vopěnka's Principle prevailed and, to this day, it is an active[1] area of research.

Returning to the more mathematical part of this section, Vopěnka's Principle, from now on VP, is a (very) large cardinal axiom based on the concept of elementary embeddings. Without further ado, let us give the definition of VP.

Definition 5.1 (Vopěnka's Principle)**.** For every proper class $\mathcal{C}$ of structures of the same signature, there exist $A, B \in \mathcal{C}$, with $A \neq B$, such that A is elementarily embeddable into B.

Regarding definability, since VP requires quantification over proper classes, it cannot be stated as a single axiom in the language of first-order ZFC set theory[2]. So, in our context, we formulate VP as the following axiom schema. For each formula $\phi(x, y)$, we have the

[1] Maybe a bit less active than machine learning.

[2] There is, however, a formulation of VP as a single axiom in NBG, the von Neumann–Bernays–Gödel set theory, that is conservative over the Vopěnka scheme (i.e., the one we are using) for first-order statements in the language of set theory (cf. [9]).

schema instance:

$$\forall x\Big(\forall y\forall z\big(\phi(x,y)\wedge\phi(x,z)\rightarrow \text{“}y \text{ and } z \text{ are structures of the same signature”}\big)\wedge$$
$$\forall\alpha\in On\exists y\big(rank(y)>\alpha\wedge\phi(x,y)\big)\rightarrow$$
$$\exists y\exists z\big(\phi(x,y)\wedge\phi(x,z)\wedge y\neq z\wedge\exists j(\text{“ } j:y\rightarrow z \text{ is elementary”})\big)\Big)$$

Consider now the following two variants of VP.

Definition 5.2. For $n\geq 0$, if κ is an infinite cardinal and Γ is one of $\boldsymbol{\Sigma_n},\boldsymbol{\Pi_n}$, then, $VP(\kappa,\Gamma)$ is the following statement: for every Γ proper class $\mathcal{C}$ of structures of the same signature σ, if both σ and the parameters of some Γ-definition of $\mathcal{C}$ belong to H_κ, then for every $B\in\mathcal{C}$, there exists an $A\in\mathcal{C}\cap H_\kappa$, such that A is elementarily embeddable into B.

For $\kappa,\mathcal{C}$ and Γ as in the definition above, if $VP(\kappa,\Gamma)$ holds, we will say that $\mathcal{C}$ *reflects* below κ.

Definition 5.3. For $n\geq 0$, if Γ is one of $\boldsymbol{\Sigma_n},\boldsymbol{\Pi_n},\Sigma_n$ or Π_n, then $VP(\Gamma)$ is the following statement: for every Γ proper class $\mathcal{C}$ of structures of the language of set theory with finitely-many additional unary relation symbols, there exist distinct $A,B\in\mathcal{C}$ such that A is elementarily embeddable into B.

Two straightforward facts are:

- VP implies that, for every particular $n\geq 0$, $VP(\boldsymbol{\Pi_n})$ (and $VP(\boldsymbol{\Sigma_n})$) holds.
- If, for every $n\geq 0$, $VP(\kappa,\boldsymbol{\Pi_n})$ (or $VP(\kappa,\boldsymbol{\Sigma_n})$) holds for a proper class of cardinals κ, then VP holds.

We will now investigate these two notions with the aim of unveiling their connection with $C^{(n)}$-cardinals.

The next theorem implies that, for $\boldsymbol{\Sigma_1}$ classes, $VP(\kappa,\boldsymbol{\Sigma_1})$ already follows from ZFC.

Theorem 5.4. If κ is an uncountable cardinal and $\mathcal{C}$ a $\boldsymbol{\Sigma_1}$ class of structures of the same signature $\sigma\in H_\kappa$ with parameters in H_κ (i.e., the parameters of some $\boldsymbol{\Sigma_1}$-definition of $\mathcal{C}$ belong to H_κ), then $VP(\kappa,\boldsymbol{\Sigma_1})$ holds.

Proof. Let κ, σ and $\mathcal{C}$ be as above. Furthermore, let $B\in\mathcal{C}$ and $\lambda>\kappa$ be a regular cardinal such that $B\in H_\lambda$. By applying the Löwenheim-Skolem theorem, we get an elementary substructure N of H_λ such that N is of cardinality less than κ, with $B\in N$, $trcl(\{\sigma\})\subseteq N$ and, for some parameter b of a $\boldsymbol{\Sigma_1}$-definition of $\mathcal{C}$, we have that $b\in N$. Note that, this is possible since $\sigma\in H_\kappa$, which implies, by definition, that $|trcl(\{\sigma\})|<\kappa$.

Now, we can apply the Mostowski collapsing lemma on N and get its transitive collapse M. Let j be the inverse of the Mostowski isomorphism, i.e., $j=\pi^{-1}:M\rightarrow N$, and let $A=\pi(B)$. Then, $A\in H_\kappa$, since $|M|=|N|<\kappa$ and both A and M are transitive with

$A \in M$. Moreover, since $trcl(\{\sigma\}) \subseteq N$, we have that $\pi(\sigma) = \sigma$ and, thus, A and B are structures of the same signature σ.

We now check that $j \restriction A : A \to B$ is an elementary embedding. Let $\phi(x)$ be a formula and $a \in A$. We have that,

$$A \models \phi(a) \Leftrightarrow M \models \phi^A(a) \Leftrightarrow N \models \phi^B(j(a)) \Leftrightarrow H_\lambda \models \phi^B(j(a)) \Leftrightarrow B \models \phi(j \restriction A(a))$$

where, for the equivalences above, we have used the following facts: for the first one, that M and A are transitive, for the second, that j is an isomorphim, for the third, that $N \prec H_\lambda$ and, for the last one, that $H_\lambda \prec_1 V$ and that $a \in dom(j \restriction A)$.

Finally, we have that $B \in \mathcal{C}$ and $b, B \in N$ with $N \models B \in \mathcal{C}$. Thus, from j we get that $M \models A \in \mathcal{C}$ and, recalling that $\mathcal{C}$ is $\mathbf{\Sigma_1}$, the formula defining $\mathcal{C}$ is upwards absolute for M. Hence, we get that $A \in \mathcal{C}$ and, in consequence, $\mathcal{C}$ reflects below κ. □

On the other hand, the assumptions $VP(\Pi_1)$ and $VP(\mathbf{\Pi_1})$ have strong consequences.

5.2 Vopěnka's Principle and supercompactness

Theorem 5.5. If $VP(\Pi_1)$ holds, then there exists a supercompact cardinal.

Proof. Suppose that $VP(\Pi_1)$ holds and let $\mathcal{C}$ be the following class of structures: $X \in \mathcal{C}$ if and only if $X = \langle V_{\lambda+2}, \in, \alpha, \lambda \rangle$, where λ is the least ordinal greater than α such that there is no $< \lambda$-supercompact cardinal up to α, i.e., if $\kappa \leq \alpha$, then κ is not γ-supercompact for some $\gamma < \lambda$. Observe that, $X \in \mathcal{C}$ if and only if $X = \langle X_0, X_1, X_2, X_3 \rangle$, where

1. X_2 is an ordinal,
2. X_3 is a limit ordinal greater than X_2,
3. $X_0 = V_{X_3+2}$,
4. $X_1 = \in \restriction X_0$,
5. $\langle X_0, X_1 \rangle$ satisfies the following:
   (a) $\forall \kappa \leq X_2$("κ is not $< X_3$-supercompact")
   (b) $\forall \mu\big(($"μ limit" $\wedge X_2 < \mu < X_3) \to \exists \kappa \leq X_2($"$\kappa$ is $< \mu$-supercompact"$)\big)$

If $\phi(x)$ is the conjunction of the five above conditions, then, by basic absoluteness results[3], $\phi(x)$ is a Π_1 formula (without parameters) that defines $\mathcal{C}$.

[3]The (general) "trick" that is used, and that is worth mentioning, is that verifying if a cardinal κ is (or is not) γ-supercompact, for some γ, can be done locally. That is, we just have to check if a normal ultrafilter exists over $\mathcal{P}_\kappa(\gamma)$; which can be done inside $V_{\gamma+5}$. With that in mind, and the fact that the satisfiability relation (for sets) is Δ_1, the conditions of (5) are Π_1-definable.

Now, for the sake of contradiction, suppose that there is no supercompact cardinal. We claim that $\mathcal{C}$ is a proper class of structures. To see this, fix $\alpha \in On$. Then, for every $\kappa \leq \alpha$, there exists a $\gamma \in On$ such that κ is not γ-supercompact and, obviously, κ is not β-supercompact for every $\beta > \gamma$. Now, for every $\kappa \leq \alpha$, let δ_κ be the least such γ above α and let $\lambda = \sup\{\delta_\kappa : \kappa \leq \alpha\}$. Then, for every $\kappa \leq \alpha$, V_λ contains witnesses of the failure of the $< \lambda$-supercompactness of κ and thus, $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \in \mathcal{C}$ with $\lambda > \alpha$.

Since $\mathcal{C}$ is a proper class and $VP(\Pi_1)$ holds, there are two distinct structures $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle$ and $\langle V_{\mu+2}, \in, \beta, \mu\rangle$, and an elementary embedding

$$j : \langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \prec \langle V_{\mu+2}, \in, \beta, \mu\rangle$$

Note that, $j(\alpha) = \beta$ and $j(\lambda) = \mu$, since α, β, λ and μ are constants of the signature of the structures. Moreover, by (a simple corollary of) Kunen's theorem, j cannot be the identity and so $\lambda < \mu$. Consequently, even though there is no $\kappa \leq \alpha$ that is $< \lambda$-supercompact, there is some $\kappa \leq \beta$ which is $< \lambda$-supercompact (otherwise, λ would be equal to μ). Hence, by the definition of λ and μ from α and β respectively, we have that $\alpha < \beta$. Let κ be the critical point of j. Then, since $\alpha < \beta$ and $j(\alpha) = \beta$, we have that $\kappa \leq \alpha$. By a well-known result of Magidor[4], we have that κ is $< \lambda$-supercompact, which is a contradiction, since $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \in \mathcal{C}$. □

Theorem 5.6. If $VP(\mathbf{\Pi_1})$ holds, then there exist a proper class of supercompact cardinals.

Proof. Fix an ordinal γ and let $\mathcal{C}$ be the class of structures of the form $\langle V_{\lambda+2}, \in, \alpha, \lambda, R_\gamma\rangle$, where R_γ is the (unary) relation $\{\delta \in On : \delta < \gamma + 1\}$, $\alpha > \gamma$ and λ is the least limit ordinal greater than α, such that no κ with $\gamma < \kappa \leq \alpha$ is $< \lambda$-supercompact. Note that, with the addition of R_γ, if we have two structures $A, B \in \mathcal{C}$ and a $j : A \prec B$, then, by elementarity, j must be the identity on all ordinals $< \gamma + 1$. Now, arguing as in the previous theorem, the class $\mathcal{C}$ is $\mathbf{\Pi_1}$ with parameter γ and if we assume that there are no supercompact cardinals above γ, then $\mathcal{C}$ is a proper class. Hence, we can apply $VP(\mathbf{\Pi_1})$ to get a contradiction as before, noting that, this time, the critical point of j is greater than γ. □

In the other direction, if $\mathcal{C}$ is a Π_1 class of structures (of the same signature), then, a question is how much supercompactness do we need in order for VP to hold for $\mathcal{C}$. Before trying to answer this, let us first introduce the following notion.

Definition 5.7. We will say that a limit ordinal λ *captures* a proper class $\mathcal{C}$ if the class of ordinal ranks of elements of $\mathcal{C}$, intersected with λ, is unbounded in λ.

A simple observation is the following.

Proposition 5.8. For $n \geq 1$, if $\mathcal{C}$ is a $\mathbf{\Pi_n}$ proper class of structures and λ a $C^{(n+1)}$-cardinal which is greater than the rank of the parameters of some $\mathbf{\Pi_n}$-definition of $\mathcal{C}$, then λ captures $\mathcal{C}$.

[4] The result, which appears in [14], states that if $j : V_\lambda \prec V_\mu$ is an elementary embedding with $cp(j) = \kappa$ and λ, μ are limit ordinals, then κ is $< \lambda$-supercompact.

Proof. Let $\alpha < \lambda$. We have to show that there exists a $\beta \in On$ such that $\alpha < \beta < \lambda$ and, for some $A \in \mathcal{C}$, the rank of A is β. The assertion

$$\exists x\big(x \in \mathcal{C} \wedge rank(x) > \alpha\big)$$

is a Σ_{n+1} sentence (since "$x \in \mathcal{C}$" is Π_n and "$rank(x) > \alpha$" is Π_1) with parameters α and the parameters of some $\boldsymbol{\Pi_n}$-definition of $\mathcal{C}$. Now, since $\mathcal{C}$ is a proper class, the statement above holds in V and, since $\lambda \in C^{(n+1)}$, it also holds in V_λ. In other words, for $\alpha < \lambda$, there is an $A \in \mathcal{C} \cap V_\lambda$, such that:

$$V_\lambda \models A \in \mathcal{C} \wedge rank(A) > \alpha$$

and, again, since $\lambda \in C^{(n+1)}$ and $rank(A) < \lambda$, we get that $A \in \mathcal{C}$ and $\alpha < rank(A) < \lambda$. □

The following proposition gives an upper bound on the supercompactness that is needed, as a partial answer to the question above.

Proposition 5.9. Let $\mathcal{C}$ be a Π_1 proper class of structures of the same signature. If there exists a cardinal κ that is $< \lambda$-supercompact, for some $\lambda \in Lim(C^{(1)})$ greater than κ that captures $\mathcal{C}$, then VP holds for $\mathcal{C}$.

Proof. Since λ captures $\mathcal{C}$ and $\lambda \in Lim(C^{(1)})$, we can find $\delta < \lambda$ and $B \in \mathcal{C} \cap V_\delta$ with $\delta \in C^{(1)}$ and $rank(B) > \kappa$. Let $j : V \prec M$ be an elementary embedding witnessing the δ-supercompactness of κ. Now, since $V_\delta = H_\delta$ (as $\delta \in C^{(1)}$) and M is closed under δ-sequences, by an easy induction we have that $V_\delta^M = V_\delta$. Thus, B, which belongs to V_δ, belongs to M and, because M is an inner model and $\mathcal{C}$ is Π_1, we have that $M \models B \in \mathcal{C}$. Moreover, by the usual arguments, the function $j \restriction B : B \to j(B)$ is an elementary embedding. Once again, by the closure of M, $j \restriction B \in M$ and so we get that

$$M \models \exists x \in \mathcal{C} \exists e\big(rank(x) < j(\kappa) \wedge \text{“}e : x \to j(B) \text{ is elementary”}\big)$$

since B and $j \restriction B$ are the witnesses for x and e respectively. By elementarity, we get that

$$\exists x \in \mathcal{C} \exists e\big(rank(x) < \kappa \wedge \text{“}e : x \to B \text{ is elementary”}\big)$$

holds, which is exactly what we wanted. □

Recall that, in this chapter, we want to present a correspondence between large cardinals and variants of VP and, ultimately, reveal the connection that joins the hierarchy of $C^{(n)}$-extendible cardinals with structural reflection phenomena. The next theorem, with the following two corollaries, completes the first step in providing the equivalence at the lowest level.

Theorem 5.10. ([5]) Let $\mathcal{C}$ be a $\boldsymbol{\Sigma_2}$ class of structures of the same signature σ. Moreover, suppose that κ is a supercompact cardinal larger than the rank of the parameters of some $\boldsymbol{\Sigma_2}$-definition of $\mathcal{C}$, with $\sigma \in V_\kappa$. Then, $VP(\kappa, \boldsymbol{\Sigma_2})$ holds.

Proof. Let κ, σ and $\mathcal{C}$ be as above. Also, let $\phi(x, y)$ be a Σ_2 formula that defines $\mathcal{C}$ and let $b \in V_\kappa$ be the parameter of that $\mathbf{\Sigma_2}$-definition of $\mathcal{C}$.

Fix $B \in \mathcal{C}$ and let λ be a $C^{(2)}$-cardinal greater than $rank(B)$. Let $j : V \prec M$ be an elementary embedding witnessing the λ-supercompactness of κ. By the usual arguments, B and $j \restriction B : B \to j(B)$ are in M. Furthermore, since $\lambda \in C^{(2)}$ and M is closed under λ-sequences, by induction, we have that $V_\lambda \in M$. Hence, by Lévy's theorem, we have that $V_\lambda \prec_1 M$. Moreover, $\sigma \in V_\kappa$ and thus, $j(\sigma) = \sigma$ and $j(B)$ is a structure of signature σ. Lastly, again by the usual arguments, $j \restriction B$ is an elementary embedding.

Now, since $B \in \mathcal{C}$ and $B, b \in V_\lambda$ and $\lambda \in C^{(2)}$, we have that $V_\lambda \vDash \phi(B, b)$. In addition, since $V_\lambda \prec_1 M$, Σ_2 formulas are upwards absolute between V_λ and M. Thus, $M \vDash \phi(B, b)$. In other words, noting that $V_\lambda \subseteq V^M_{j(\kappa)}$, we have that

$$M \vDash \exists x \exists e \big(x \in V_{j(\kappa)} \wedge \phi(x, b) \wedge \text{“}e : x \to j(B) \text{ is elementary”}\big)$$

with the obvious witnesses; B and $j \restriction B$. By elementarity, we have exactly what we wanted, i.e., $\mathcal{C}$ reflects below κ. □

We immediately get the following two corollaries.

Corollary 5.11. The following are equivalent:

1. $VP(\Pi_1)$.
2. $VP(\kappa, \mathbf{\Sigma_2})$, for some κ.
3. There exists a supercompact cardinal.

Proof. $2 \Rightarrow 1$ is trivial. $1 \Rightarrow 3$ is from Theorem 5.5 and $3 \Rightarrow 2$ is from the previous theorem. □

Corollary 5.12. The following are equivalent:

1. $VP(\mathbf{\Pi_1})$.
2. $VP(\kappa, \mathbf{\Sigma_2})$, for a proper class of cardinals κ.
3. There exists a proper class of supercompact cardinals.

Proof. Again, $2 \Rightarrow 1$ is trivial. $1 \Rightarrow 3$ is from Theorem 5.6 and $3 \Rightarrow 2$ is from the previous theorem. □

We next proceed with a useful theorem of Magidor, characterizing the least cardinal that reflects classes of structures of the form $\langle V_\alpha, \in \rangle$ as supercompact.

Theorem 5.13. ([14]) If μ is the least cardinal that reflects the Π_1 proper class $\mathcal{C}$ of structures of the form $\langle V_\beta, \in \rangle$, then μ is supercompact.

Proof. Let $\lambda > \mu$ be a singular $C^{(2)}$ cardinal. By the hypothesis, there is an ordinal $\alpha < \mu$ and an elementary embedding $j : V_{\alpha+1} \prec V_{\lambda+1}$. Let $cp(j) = \kappa$. We claim that $\kappa < \alpha$.

Towards a contradiction, suppose that $\kappa = \alpha$. Then, $V_{\alpha+1} \models$ "α is regular", since α is the critical point of j. By elementarity, and since $j(\alpha) = \lambda$, $V_{\lambda+1} \models$ "λ is regular", which in turn implies that λ is indeed a regular cardinal, contrary to our assumption.

Now, since α is a limit ordinal, by Lemma 2 of [14], we have that κ is $< \alpha$-supercompact and, thus, $V_\alpha \models$ "κ is supercompact". By the usual arguments, $j \restriction V_\alpha : V_\alpha \to V_\lambda$ is an elementary embedding and, by elementarity, we get that $V_\lambda \models$ "$j(\kappa)$ is supercompact". Moreover, recalling that the property of being supercompact is Π_2 expressible, we get that $j(\kappa)$ is indeed supercompact. By Theorem 5.10, $j(\kappa)$ reflects $\mathcal{C}$, and thus, by the minimality of μ, we have that $j(\kappa) \geq \mu$. Observe that, if $j(\kappa) = \mu$, then, the proof is completed.

So, assume for the sake of contradiction, that $j(\kappa) > \mu$. Furthermore, note that the property "μ reflects $\mathcal{C}$" is Π_2 and, by Theorem 3.6, we have that $V_{j(\kappa)} \prec_2 V$. Hence, $V_{j(\kappa)} \models$ "μ reflects $\mathcal{C}$". By elementarity, we have that, for some $\gamma < \kappa$, $V_\kappa \models$ "γ reflects $\mathcal{C}$" and, once again by elementarity, we get that $V_{j(\kappa)} \models$ "γ reflects $\mathcal{C}$", since $j(\gamma) = \gamma$. This in turn implies that γ does indeed reflect $\mathcal{C}$, contradicting the minimality of μ. □

Using the previous theorem and Theorem 5.10, we easily get the following corollary.

Corollary 5.14. The following are equivalent:

1. κ is the first supercompact cardinal.
2. κ is the least cardinal for which $VP(\kappa, \mathbf{\Sigma_2})$ holds.
3. κ is the least cardinal that reflects the Π_1 class of structures of the form $\langle V_\alpha, \in \rangle$, for $\alpha \in On$.

Proof. If κ is a supercompact cardinal, then, by Theorem 5.10, $VP(\kappa, \mathbf{\Sigma_2})$ holds. Hence, κ reflects the class of structures of the form $\langle V_\alpha, \in \rangle$, for $\alpha \in On$. By the preceding theorem, $(1), (2)$ and (3) are equivalent. □

With the next theorem, we finally complete the first step in building the aforementioned correspondence.

Theorem 5.15. For a cardinal κ, $VP(\kappa, \mathbf{\Pi_1})$ holds if and only if either κ is a supercompact cardinal or a limit of supercompact cardinals.

Proof. $(\Leftarrow)$: First, note that reflecting $\mathbf{\Pi_1}$ classes of structures is closed under limits of supercompact cardinals: let $\mathcal{C}$ be a $\mathbf{\Pi_1}$ class of structures and suppose that for $\gamma \in On$, $\langle a_\xi : \xi < \gamma \rangle$ is a sequence of supercompact cardinals. Let κ be the limit of the sequence and $B \in \mathcal{C}$. Furthermore, assume that the parameter, b, of some $\mathbf{\Pi_1}$-definition of $\mathcal{C}$ is in V_κ. Since κ is a limit, there is a $\xi < \gamma$ such that $b \in V_{a_\xi}$. From Theorem 5.10, there is an $A \in \mathcal{C} \cap V_{a_\xi} \subseteq V_\kappa$ that is elementarily embeddable into B.

So, if κ is supercompact or a limit of supercompact cardinals, from Theorem 5.10 we get that κ reflects all $\mathbf{\Pi_1}$ classes.

$(\Rightarrow)$: Suppose, towards a contradiction, that κ is neither a supercompact cardinal, nor a limit of supercompact cardinals. Then, there is a $\gamma < \kappa$ such that, there is no supercompact cardinal in between γ and κ. With that in mind, the proof now is similar to the proof of Theorem 5.6. □

In order to extend the aforementioned correspondence further, we have to ascend higher in the large cardinal hierarchy. In particular, we now focus our attention on $C^{(n)}$-extendible cardinals.

5.3 Vopěnka's Principle and $C^{(n)}$-extendibility

Recall that, from Theorem 5.10, if κ is supercompact, then $VP(\kappa, \mathbf{\Sigma_2})$ holds. Going one step further, as we will see, if κ is extendible, then κ has stronger reflection properties; in particular, $VP(\kappa, \mathbf{\Sigma_3})$ holds. In fact, more generally for $n \geq 1$, the following theorem provides a lower bound of reflection that a $C^{(n)}$-extendible cardinal possesses.

Theorem 5.16. For $n \geq 1$, let $\mathcal{C}$ be a $\mathbf{\Sigma_{n+2}}$ class of structures of the same signature σ. Moreover, suppose that κ is a $C^{(n)}$-extendible cardinal larger than the rank of the parameters of some $\mathbf{\Sigma_{n+2}}$-definition of $\mathcal{C}$, with $\sigma \in V_\kappa$. Then, $VP(\kappa, \mathbf{\Sigma_{n+2}})$ holds.

Proof. Fix a natural number $n \geq 1$ and let κ, σ and $\mathcal{C}$ be as above. Moreover, fix a Σ_{n+2} formula $\phi(y, z)$ such that, for some set $b \in V_\kappa$,

$$A \in \mathcal{C} \Leftrightarrow \phi(A, b)$$

Let $B \in \mathcal{C}$ and $\lambda \in C^{(n+2)}$, with $\lambda > \kappa$ and $B \in V_\lambda$. Since $\mathcal{C}$ is $\mathbf{\Sigma_{n+2}}$, we have that

$$V_\lambda \models \phi(B, b)$$

Now, κ is $C^{(n)}$-extendibe and so there is an elementary embedding $j : V_\lambda \prec V_\mu$ with $cp(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. As usual, noting that μ is a limit ordinal, $j \restriction B : B \to j(B)$ is an elementary embedding that belongs to V_μ.

We now claim that $V_{j(\kappa)} \prec_{n+2} V_\mu$. From Proposition 4.10, κ belongs to $C^{(n+2)}$ and thus $V_\kappa \prec_{n+2} V_\lambda$. In addition, recall that, for every fixed $m \geq 0$, the satisfaction relation restricted to Σ_m formulas, $\models_m$, is formalizable and so we have the following:

$$V_\lambda \models \forall x \in V_\kappa \forall \theta(x) \in \Sigma_{n+2}(V_\kappa \models \theta(x) \leftrightarrow \models_{n+2} \theta(x))$$

By elementarity, we have that

$$V_\mu \models \forall x \in V_{j(\kappa)} \forall \theta(x) \in \Sigma_{n+2}(V_{j(\kappa)} \models \theta(x) \leftrightarrow \models_{n+2} \theta(x))$$

or, in other words, that $V_{j(\kappa)} \prec_{n+2} V_\mu$.

Moreover, it is easy to check that $V_\lambda \prec_{n+1} V_{j(\kappa)}$. For, suppose that $\phi(x)$ is a Σ_{n+1} formula and $a \in V_\lambda$. Then, since $V_\lambda \prec_{n+2} V_\kappa$ and $V_\kappa \prec V_{j(\kappa)}$, we have that

$$V_\lambda \vDash \phi(a) \Leftrightarrow V_\kappa \vDash \phi(a) \Leftrightarrow V_{j(\kappa)} \vDash \phi(a)$$

In consequence, we get that $V_\lambda \prec_{n+1} V_\mu$ and, since $V_\lambda \vDash \phi(B,b)$, we have that $V_\mu \vDash \phi(B,b)$.

Lastly, to conclude the proof, we make the following observation:

$$V_\mu \vDash \exists x \in V_{j(\kappa)} \exists e\big(x \in \mathcal{C} \wedge \text{“}e : x \to j(B) \text{ is elementary”}\big)$$

since it is true for B and $j \restriction B$, both of which belong to V_μ. Thus, by elementarity,

$$V_\lambda \vDash \exists x \in V_\kappa \exists e\big(x \in \mathcal{C} \wedge \text{“}e : x \to B \text{ is elementary”}\big)$$

Remembering that $\lambda \in C^{(n+2)}$, there is indeed an $A \in V_\kappa$ and an elementary embedding $e : A \prec B$, which is what we wanted. □

Now, recall the definition of $C^{(n)+}$-extendible cardinals (Definition 4.19). In the same vein as Theorem 5.5, we have the following theorem.

Theorem 5.17. For $n \geq 1$, if $VP(\Pi_{n+1})$ holds, then there exists a $C^{(n)+}$-extendible cardinal.

Proof. Suppose, for the sake of contradiction, that there are no $C^{(n)+}$-extendible cardinals. Then, for every $\alpha \in On$, there is a $\lambda \in C^{(n)}$, with $\lambda > \alpha$, such that α is not λ-$C^{(n)+}$-extendible. Moreover, it is easy to see that, for every $\alpha \in On$, if α is *not* λ-$C^{(n)+}$-extendible, for some $\lambda \in C^{(n)}$, then α is *not* λ'-$C^{(n)+}$-extendible for every $\lambda' > \lambda$ since, if this was not the case, if $j : V_{\lambda'} \prec V_{j(\lambda')}$ was a witness to the λ'-$C^{(n)}$-extendibility of κ, then, $j \restriction V_\lambda$ would witness the λ-$C^{(n)}$-extendibility of κ; a contradiction.[5] Hence, the class function

$$F(\alpha) = \text{ “the least } \lambda \in C^{(n+1)} \text{ greater than } \alpha \text{ such that } \alpha \text{ is not } \lambda\text{-}C^{(n)+}\text{-extendible ”}$$

is well-defined for every $\alpha \in On$.

Consider now the ordinal class $\mathcal{D} = \{\eta > 0 : \forall \alpha < \eta(F(\alpha) < \eta)\}$. It is easy to see that $\mathcal{D}$ is a club proper class of ordinals and, moreover, that $\mathcal{D} \subseteq C^{(n+1)}$. We claim that, F is Π_{n+1}. For, suppose that $\alpha \in On$. Then, $\lambda = F(\alpha)$ if and only if the following conditions are satisfied:

1. $\lambda \in C^{(n+1)}$
2. $\alpha < \lambda$
3. $\forall \beta > \lambda\big(\beta \in C^{(n)} \to V_\beta \vDash \text{“}\alpha \text{ is not } \lambda\text{-}C^{(n)+}\text{-extendible”}\big)$
4. $V_\lambda \vDash \forall \lambda' > \alpha\big(\lambda' \in C^{(n+1)} \to \text{“}\alpha \text{ is } \lambda'\text{-}C^{(n)+}\text{-extendible”}\big)$

[5]Note that this has nothing to do with the notion of $C^{(n)+}$-extendibility but is, rather, a simple property of usual extendibility.

For the fourth condition above, recall that, for $\alpha \in On$ and $\lambda' > \alpha$, the statement "α is λ'-$C^{(n)+}$-extendible" is Σ_{n+1}. This suggests that, in order to verify that λ is indeed the least $C^{(n+1)}$ cardinal for which "α is not λ-$C^{(n)+}$-extendible", it suffices to check inside V_λ that α is λ'-$C^{(n)+}$-extendible, for every other $\lambda' \in C^{(n+1)}$ with $\lambda' > \alpha$. Similarly, if the third condition above holds, then α is not a λ-$C^{(n)+}$-extendible cardinal: for, suppose that α was λ-$C^{(n)+}$-extendible. Then, pick a $\lambda' \in C^{(n+1)}$ greater than λ. Remembering that the property of being λ-$C^{(n)+}$-extendible is Σ_{n+1}, we get that $V_{\lambda'} \models$ "α is λ-$C^{(n)+}$-extendible". But since λ' belongs in $C^{(n+1)}$, we also have that $\lambda' \in C^{(n)}$. Thus, we get a contradiction to (3). With that being said, taking the conjunction of the four conditions above we get that F is Π_{n+1} and, thus, $\mathcal{D}$ is Π_{n+1}. Let $\phi(x)$ be a Π_{n+1} formula defining $\mathcal{D}$.

Now, for every $\alpha \in On$, let us use the notation λ_α for the least limit point of $\mathcal{D}$ greater than α. Once again, the class function $\alpha \mapsto \lambda_\alpha$ is Π_{n+1} since, $x = \lambda_\alpha$ if and only if the following conditions hold:

1. $x \in On$ with $x > \alpha$
2. $x \in \mathcal{D}$
3. $V_x \models \forall\beta\exists\gamma\big(\gamma > \beta \wedge \phi(\gamma)\big)$
4. $V_x \models \forall\beta\big(\beta > \alpha \to \exists\gamma < \beta\forall\eta(\gamma < \eta < \beta \to \neg\phi(\eta))\big)$

All this work was done so that we can, finally, use the following proper class of structures $\mathcal{C} = \{A_\alpha : \alpha \in On\}$, where

$$A_\alpha = \langle V_{\lambda_\alpha}, \in, \alpha, \lambda_\alpha, \mathcal{D} \cap (\alpha + 1)\rangle$$

As one would expect, $\mathcal{C}$ is a Π_{n+1} proper class: we have that $X \in \mathcal{C}$ if and only if $X = \langle X_0, X_1, X_2, X_3, X_4\rangle$, where

1. $X_2 \in \mathcal{D}$
2. $X_3 = \lambda_{X_2}$
3. $X_0 = V_{X_3}$
4. $X_1 = \in\restriction X_0$
5. $X_4 = \mathcal{D} \cap (X_2 + 1)$

Thus, from $VP(\Pi_{n+1})$, there exist $\alpha \neq \beta$ such that A_α and A_β belong to $\mathcal{C}$ and there exists an elementary embedding $j : A_\alpha \prec A_\beta$. From the signature of the structures A_α and A_β, we have that $\alpha < \beta$ and $j(\alpha) = \beta$. Hence, j is not the identity and if $cp(j) = \kappa$, then $\kappa \leq \alpha$.

We now claim that $\kappa \in \mathcal{D}$; for, otherwise, let $\gamma = \sup(\mathcal{D} \cap \kappa) < \kappa$. Moreover, let $\delta \in \mathcal{D}$ be the least ordinal greater than γ such that $\delta < \lambda_\alpha$. From basic absoluteness results, and since $j(\gamma) = \gamma$, we have that $j(\delta) = \delta$. But then, $j \restriction V_{\delta+2} : V_{\delta+2} \prec V_{\delta+2}$ is a nontrivial elementary embedding, contradicting Kunen's Theorem.

Now, since $\mathcal{D} \subseteq C^{(n+1)}$, we have that $\lambda_\alpha \in C^{(n+1)}$ and thus, that $V_{\lambda_\alpha} \models \kappa \in \mathcal{D}$. By elementarity, $V_{\lambda_\beta} \models j(\kappa) \in \mathcal{D}$ and, once again, since $\lambda_\beta \in C^{(n+1)}$, it follows that $j(\kappa) \in \mathcal{D}$. Hence, by the definition of $\mathcal{D}$, we have that $F(\kappa) < j(\kappa)$ and, moreover, since $\lambda_\alpha \in \mathcal{D}$, we have that $F(\kappa) < \lambda_\alpha$. Furthermore, since $F(\kappa) \in C^{(n+1)}$, we have that $V_{\lambda_\alpha} \models F(\kappa) \in C^{(n+1)}$ and, by elementarity, $V_{\lambda_\beta} \models j(F(\kappa)) \in C^{(n+1)}$, which in turn implies that $j(F(\kappa))$ is indeed a $C^{(n+1)}$ cardinal.

Lastly, once again, by the usual arguments, $j \restriction V_{F(\kappa)} : V_{F(\kappa)} \to V_{j(F(\kappa))}$ is an elementary embedding with $cp(j \restriction V_{F(\kappa)}) = cp(j) = \kappa$, witnessing the $F(\kappa)$-$C^{(n)+}$-extendibility of κ; a contradiction to the definition of F. □

Combining Theorems 5.16 and 5.17, we have the following.

Corollary 5.18. For $n \geq 1$, the following are equivalent.

1. $VP(\Pi_{n+1})$.
2. $VP(\kappa, \mathbf{\Sigma_{n+2}})$, for some κ.
3. There exists a $C^{(n)}$-extendible cardinal.
4. There exists a $C^{(n)+}$-extendible cardinal.

Proof. $1 \Rightarrow 4$ is Theorem 5.17, $4 \Rightarrow 3$ and $2 \Rightarrow 1$ are immediate and, $3 \Rightarrow 2$ follows from Theorem 5.16 □

Summing up the results of this section, the promised correspondence between levels of VP and $C^{(n)}$-extendible cardinals can now be stated in the form of the following corollary.

Corollary 5.19. The following (schemata) are equivalent[6]

1. $VP(\Pi_n)$, for every $n \geq 1$.
2. $VP(\kappa, \mathbf{\Sigma_n})$, for a proper class of cardinals κ, and for every $n \geq 1$.
3. VP.
4. For every $n \geq 1$, there exists a $C^{(n)}$-extendible cardinal.
5. For every $n \geq 1$, there exists a $C^{(n)+}$-extendible cardinal.

Proof. $1, 2, 4$ and 5 are equivalent from Corollary 5.18. 3 implies 1 and 2 implies 3. □

Taking into account the preceding corollary, we can now update the area in between extendible cardinals and Vopěnka's principle in the large cardinal figure that we presented in the beginning:

[6]The equivalence of $(2), (3)$ and (5) was already proved in [5].

⋮

|

Vopěnka principle

|

⋮

|

$C^{(n+1)}$-extendible

|

$C^{(n)}$-extendible

|

⋮

|

$C^{(1)}$-extenbile = extendible

|

⋮

We end this section by providing a characterization of $C^{(n)}$-extendible cardinals in terms of reflection of classes of structures.

Theorem 5.20. If $n \geq 1$ and μ is the least cardinal that reflects all Π_{n+1} proper classes of structures of the same signature, then μ is $C^{(n)+}$-extendible.

Proof. Towards a contradiction, suppose that μ is not $C^{(n)+}$-extendible. Then, there are neither $C^{(n)}$-extendible, nor $C^{(n)+}$-extendible cardinals below μ, since by Theorem 5.16, they would contradict the minimality of μ.

Let $\mathcal{C}$ be the class of structures of the form $\langle V_\xi, \in, \lambda, \alpha, C^{(n)} \cap \xi \rangle$, where $\alpha < \lambda < \xi$ and the following hold:

1. $\lambda \in C^{(n)}$
2. $\xi \in Lim(C^{(n)})$
3. $cf(\xi)$ is uncountable
4. $\forall \beta < \xi \forall \mu \big(\exists j (j : V_\lambda \prec V_\mu \wedge cp(j) = \alpha \wedge j(\alpha) = \beta) \rightarrow \exists j' \exists \mu' (j' : V_\lambda \prec V_{\mu'} \wedge \mu' < \xi \wedge cp(j') = \alpha \wedge j'(\alpha) = \beta \wedge V_\xi \models \mu' \in C^{(n)}) \big)$
5. λ witnesses that no ordinal less than or equal to α is λ-$C^{(n)+}$-extendible

By examing the defining complexity of the above conditions, it follows that $\mathcal{C}$ is a Π_{n+1} proper class of structures and thus, μ reflects $\mathcal{C}$. So, for $B = \langle V_\xi, \in, \lambda, \mu, C^{(n)} \cap \xi \rangle \in \mathcal{C}$, there exists $A = \langle V_{\xi'}, \in, \lambda', \alpha', C^{(n)} \cap \xi' \rangle \in \mathcal{C}$, with $rank(A) < \mu$ and $A \neq B$, and an elementary embedding $j : A \prec B$.

Let $cp(j) = \kappa$ and suppose, towards a contradiction, that $\kappa \notin C^{(n)}$. Let $\gamma = \sup(C^{(n)} \cap \kappa)$. Moreover, let $\delta \in C^{(n)}$ be the least ordinal such that $\gamma < \delta < \xi'$. Since $j(\gamma) = \gamma$, we have that $j(\delta) = \delta$, which leads to $j \restriction V_{\delta+2} : V_{\delta+2} \to V_{\delta+2}$ being elementary, contradicting Kunen's Theorem.

Additionally, if $j^m(\kappa) < \xi'$ for all $m \in \omega$, then, since ξ' has uncountable cofinality, $\sup\{j^m(\kappa) : m \in \omega\} < \xi'$ and thus belongs in $V_{\xi'}$, contradicting once again Kunen's Theorem. Consequently, there is a unique $m \in \omega$ such that, $j^m(\kappa) < \xi' \leq j^{m+1}(\kappa)$.

We now claim that, there exists an elementary embedding $e : V_{\lambda'} \prec V_\eta$, for some $\eta \in C^{(n)}$, with $cp(e) = \kappa$ and $e(\kappa) = j^{m+1}(\kappa)$. We proceed by induction on $i \leq m$. For the base case, take $e = j \restriction V_{\lambda'}$. Now, suppose that it is true for an $i < m$. We have that $j^{i+1}(\kappa) < \xi'$, and by (4) above, there exist j' and η' such that $j' : V_{\lambda'} \to V_{\eta'}$ is an elementary embedding with $\eta' < \xi'$, $cp(j') = \kappa$, $j'(\kappa) = j^{i+1}(\kappa)$ and $V_{\xi'} \vDash \eta' \in C^{(n)}$. By the elementarity of j, we have that $V_\xi \vDash j(\eta') \in C^{(n)}$, and since, from (2), $\xi \in C^{(n)}$, it follows that $j(\eta') \in C^{(n)}$. Now, it is easy to see that the composition $e = j \circ j' : V_{\lambda'} \to V_{j(\eta')}$ is an elementary embedding with $cp(e) = \kappa$ and $e(\kappa) = j^{i+2}(\kappa)$, as desired.

Lastly, since $\kappa, \xi', \xi \in C^{(n)}$, we have that $j^{m+1}(\kappa) \in C^{(n)}$. Hence, e witnesses the λ'-$C^{(n)+}$-extendibility of κ, yielding a contradiction of (5). □

Remark. Recall that, from Corollary 5.18, the existence of a $C^{(n)}$-extendible is equivalent to that of a $C^{(n)+}$-extendible cardinal. Shedding a bit more light on that, it is now easy to see that the least $C^{(n)}$-extendible cardinal is also $C^{(n)+}$-extendible. For, suppose that κ is the least $C^{(n)}$-extendible cardinal. Then, by Theorem 5.16, κ reflects all $\boldsymbol{\Sigma_{n+2}}$ proper classes of structures. Now, assume towards a contradiction, that there is a cardinal $\mu < \kappa$ which reflects all $\boldsymbol{\Sigma_{n+2}}$ classes of structures. In particular, μ reflects all Π_{n+1} proper classes. Thus, by the previous theorem, there is a cardinal $\lambda \leq \mu$ which is $C^{(n)+}$-extendible; a contradiction. Hence, κ is the least cardinal with that property and therefore, κ is $C^{(n)+}$-extendible.

In the same spirit as Corollary 5.14, we have that:

Corollary 5.21. For $n \geq 1$, the following are equivalent:

1. κ is the first $C^{(n)}$-extendible cardinal.
2. κ is the least cardinal for which $VP(\kappa, \boldsymbol{\Sigma_{n+2}})$ holds.
3. κ is the least cardinal that reflects all Π_{n+1} proper classes of structures of the form $\langle V_\alpha, \in, A\rangle$, where A is a unary predicate.

Proof. The proof follows easily from Theorems 5.16 and 5.20. □

Lastly, just as in the case of supercompact cardinals (Theorem 5.15), we give a characterization of $VP(\boldsymbol{\Pi_{n+1}})$ in terms of $C^{(n)}$-extendible cardinals.

Theorem 5.22. For a cardinal κ, $VP(\kappa, \boldsymbol{\Pi_{n+1}})$ holds if and only if κ is a $C^{(n)}$-extendible cardinal or a limit of $C^{(n)}$-extendible cardinals.

Proof. For the $(\Leftarrow)$ direction, using similar arguments as in the proof of Theorem 5.15, we have that the property of reflecting $\mathbf{\Pi_{n+1}}$ classes of structures is closed under limits. Thus, if κ is a $C^{(n)}$-extendible or a limit of $C^{(n)}$-extendible cardinals, by Theorem 5.16, it reflects all $\mathbf{\Pi_{n+1}}$ proper classes of structures.

For the $(\Rightarrow)$ direction, suppose, for the sake of contradiction, that $VP(\kappa, \mathbf{\Pi_{n+1}})$ holds and κ is neither $C^{(n)}$-extendible, nor a limit of $C^{(n)}$-extendible cardinals. Then, there is an ordinal $\eta < \kappa$ such that there is no $C^{(n)}$-extendible cardinal in between η and κ. The idea now is to find a $C^{(n)}$-extendible cardinal in between η and κ, yielding a contradiction.

Now, let $\mathcal{C}$ be the class of structures of the form $\langle V_\xi, \in, \lambda, \alpha, C^{(n)} \cap \xi, R_\eta \rangle$, where $\eta < \alpha < \lambda < \xi$, R_η is the unary relation $\{\delta \in On : \delta < \eta + 1\}$ and, moreover, the conditions $1 - 4$ of the proof of Theorem 5.20 are satisfied, as well as,

5. λ witnesses that no ordinal less than or equal to α *and greater than* η is λ-$C^{(n)}$-extendible.

Observe that $\mathcal{C}$ is also $\mathbf{\Pi_{n+1}}$ (with η as a parameter). With that being said, the proof now is similar to that of Theorem 5.20. □

With that theorem we complete this chapter and, in turn, this thesis. It should be mentioned that what we presented in the last two chapters was the starting point of the program of structural reflection that we mentioned in the introduction. Very briefly, the aim of this program is to justify large cardinal axioms in terms of some form of reflection principles. So far, the program has been undoubtedly succesful, since it has been shown that various large cardinals, in several areas of the large cardinal hierarchy, can be characterized in terms of some form of reflection (an example is exactly what we presented; the correspondence between $C^{(n)}$-extendible cardinals and fragments of Vopěnka's Principle). For further information regarding the structural reflection program, the interested reader may consult [3], [6] and [7].

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