Upper Bounds on the number of embeddings of minimally rigid graphs

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ABSTRACT

In graph theory, a rigid graph is a graph that has a finite number of embeddings in $\mathbb{R}^d$ up to rigid motions, with respect to a set of edge length constraints. An embedding of graph in $\mathbb{R}^d$ is an assignment of vertices to points in $\mathbb{R}^d$, which also induces a set of edge lengths that correspond to the distances between the connected vertices. An important class of rigid graphs is the class of minimally rigid graphs. A minimally rigid graph, is a graph that is rigid and has the property that the removal of any edge yields a graph that is not rigid. It is a major open problem to find tight upper bounds on the number of the embeddings in $\mathbb{R}^d$. For a long period, only the trivial bound of $\mathcal{O}(2^{d\cdot|V|})$ was known on the number of embeddings, that is derived from the direct application of Bézout's Theorem. In [3], the bound was improved for $d \geq 5$, using matrix permanents. Recently in [5], the asymptotic bound was improved in all dimension. In the special case of $d = 2$, the asymptotic upper bound was improved to $\mathcal{O}(3.7764^{|V|})$.

It is known that the number of solutions of a well-constrained algebraic system is related to the number of embeddings. In particular, the number of the complex solutions of such an algebraic system extends the notion of real embeddings in the complex space, allowing us to bound the complex solutions, using tools from the complex algebraic geometry. In this thesis, by counting outdegree-constrained orientations of a graph that are related to the algebraic bounds [3], we improve the existing upper bounds, for the class of minimally rigid graphs, on the number of embeddings.

ΣΥΝΟΨΗ

Στη θεωρία γραφημάτων (γράφων), ένα άκαμπτο γράφημα είναι ένα γράφημα που έχει πεπερασμένο αριθμό εμβυθίσεων στο $\mathbb{R}^d$, ως προς τις Ευκλείδιες κινήσεις, για δεδομένα μήκη ακμών. Η εμβύθιση γραφήματος στο $\mathbb{R}^d$ είναι μια ανάθεση των κορυφών σε σημεία στο $\mathbb{R}^d$, η οποία δημιουργεί ένα σύνολο με μήκη ακμών που αντιστοιχούν στις αποστάσεις μεταξύ των συνδεδεμένων κορυφών. Μια σημαντική κλάση άκαμπτων γραφημάτων είναι η κλάση των ελαχιστικώς άκαμπτων γραφημάτων. Ένα ελαχιστικώς άκαμπτο γράφημα, είναι ένα γράφημα που είναι άκαμπτο και έχει την ιδιότητα ότι η αφαίρεση οποιασδήποτε ακμής του, δίνει ένα γράφημα που δεν είναι άκαμπτο. Ένα σημαντικό ανοιχτό πρόβλημα είναι η εύρεση άνω φραγμάτων στον αριθμό τον εμβυθίσεων στο $\mathbb{R}^d$. Για ένα μεγάλο χρονικό διάστημα, μόνο το άνω φράγμα $\mathcal{O}(2^{d\cdot|V|})$ ήταν γνωστό στον αριθμό των εμβυθίσεων, που προέρχεται από την άμεση εφαρμογή του θεωρήματος του Bézout. Στο [3], το φράγμα βελτιώθηκε για $d \geq 5$, χρησιμοποιώντας τους permanent πίνακες. Πρόσφατα στο [5], το ασυμπτωτικό άνω φράγμα βελτιώθηκε για κάθε διάσταση. Στην ειδική περίπτωση του $d = 2$, το ασυμπτωτικό άνω φράγμα βελτιώθηκε σε $\mathcal{O}(3.7764^{|V|})$.

Είναι γνωστό ότι ο αριθμός των λύσεων ενός τετράγωνου αλγεβρικού συστήματος σχετίζεται με τον αριθμό των εμβυθίσεων. Συγκεκριμένα, ο αριθμός των μιγαδικών λύσεων ενός τέτοιου αλγεβρικού συστήματος επεκτείνει την έννοια των πραγματικών εμβυθίσεων στον μιγαδικό χώρο, επιτρέποντάς μας να φράξουμε τις μιγαδικές λύσεις χρησιμοποιώντας εργαλεία από τη μιγαδική αλγεβρική γεωμετρία.

Σε αυτή την διπλωματική, μετρώντας τους outdegree-περιορισμένους προσανατολισμούς ενός γραφήματος που σχετίζονται με τα αλγεβρικά φράγματα [3], βελτιώνουμε τα υπάρχοντα άνω φράγματα, για την κλάση των ελαχιστικώς άκαμπτων γραφημάτων, στον αριθμό των εμβυθίσεων.

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CONTENTS

CHAPTER 1

INTRODUCTION

Rigidity theory studies the properties of graphs that can have rigid embeddings in a specified embedding space. Besides being a mathematical area with significant research interest, it has also received much attention due to its applications in molecular biology [14], robotics [22], and architecture [1, 10]. Let $G = (V, E)$ be a simple undirected graph, i.e. it does not contain multi-edges and self-loops. Let also $\mathbb{E}^d$ be a d-dimensional Euclidean space. An embedding of G into $\mathbb{E}^d$, is an assignment of the vertices to points in $\mathbb{E}^d$. It is clear that we can have infinite such embeddings, since there are no constraints, by placing the vertices anywhere on the space.

In rigidity theory, a graph embedding is said to be *rigid* in $\mathbb{R}^d$ if and only if it admits a finite number of embeddings, up to rigid motions in $\mathbb{R}^d$, i.e. rotations and translations, while preserving the given edge lengths. Otherwise, the graph is called *flexible*. A graph is called *generically rigid* if it is rigid for almost all its embeddings. A graph G is called *generically minimally rigid* if it is *generically rigid* and any edge deletion deprives G of its *rigidity* property.

Minimally rigid graphs have received a considerable amount of attention in the past, especially in the case of $d = 2$ and $d = 3$, which are equivalent to *Laman graphs* and *Geiringer graphs* (as it is called in [11]), respectively. Due to Maxwell, there is a necessary condition for a graph to be rigid in $\mathbb{R}^d$. In particular, if a graph $G = (V, E)$ is minimally rigid, then it holds that $|E| = d \cdot |V| - \binom{d+1}{2}$, and for every subgraph $G' = (V', E')$ of G it should hold that $|E'| \leq d \cdot |V'| - \binom{d+1}{2}$ [15]. For example, in the case of $d = 2$, a *minimally rigid* graph with n vertices, should have exactly $2n - 3$ edges, and every subgraph of it with n' vertices, should have at most $2n' - 3$ edges. Note that for the *Laman graphs*, the Maxwell is condition is also sufficient, which provides a full characterization for the *minimally rigid* graphs in $\mathbb{R}^2$. On the contrary, the Maxwell condition is not sufficient for the *minimally rigid* graphs in $\mathbb{R}^3$ (see Figure 1.1 for the counter example).

From an algebraic point of view, an embedding of a graph corresponds to a solution of an algebraic system that is defined by the given edge length constraints. Thus, the number of embeddings of a *minimally rigid* graph in $\mathbb{R}^d$ is equal to the number of the real solutions of the corresponding algebraic system. The complex solutions of the same system introduces the notion of an embedding to the complex space, and this allows us to use tools from the complex algebraic geometry, for example, the Bézout's theorem.

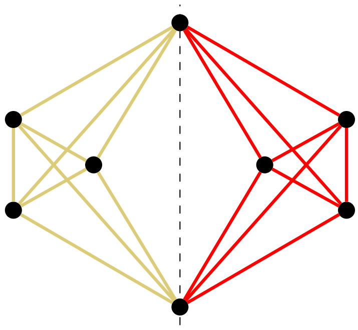

Figure 1.1: The double-banana graph satisfies Maxwell's condition in $\mathbb{R}^3$, but is not rigid since its two rigid components (red and yellow) rotate around the dashed axis.

Regarding our algebraic systems, it is clear that the number of complex solutions bound from above the number of real solutions, therefore we can work on the complex space.

An open question on rigid graph theory is to obtain tight upper bounds on the maximal number of embeddings of *minimally rigid graphs* in $\mathbb{R}^d$, depending on the number of vertices. The direct application of Bézout's theorem to the algebraic system, gives a trivial bound of $\mathcal{O}(2^{d\cdot|V|})$. Notice that, for the *Laman graphs*, the bound is of $\mathcal{O}(4^n)$ solutions. In [8], Borcea and Streinu improved the exact upper bound compared to the trivial Bézout bound, however they did not manage to improve it asymptotically. Their work yielded a bound of

$$2 \cdot \prod_{m=0}^{|V|-d-2} \frac{\binom{|V|-1+m}{|V|-d-1-m}}{\binom{2m+1}{m}}$$

embeddings. The result is based on determinantal equations (and inequalities) of the Cayley-Menger matrix [7], and a theorem on the degree of determinantal varieties [13].

Let G be a minimally rigid graph in $\mathbb{R}^d$ and $\mathcal{P}$ be its corresponding algebraic system that counts its embeddings. There is a generalization of Bézout's theorem to multi-homogeneous polynomials, called multi-homogeneous Bézout [18]. A d out-degree constrained orientation of a graph (or simply d-orientation), is an assignment of direction to every edge of G, such that every vertex has out-degree d and every edge has a direction. In [3], a relation between the multi-homogeneous Bézout of $\mathcal{P}$ and the out-degree constrained orientations of G was proven, that allows us to count orientations, in order to improve the upper bounds on the number of embeddings. The asymptotic upper bounds for $d \geq 5$ was also improved, by using the Brégman-Minc permanent bound. In [5], by bounding the number of orientations of a graph, the asymptotic order of the number of embeddings was improved for $d \geq 2$. For the case of $d = 2$, the bound is of $\mathcal{O}(3.77^n)$ solutions. This work led to the most recent bounds in all dimension ≥ 2, for the number of embeddings of *minimally rigid* graphs.

Our Contribution: In this thesis, we manage to improve the asymptotic upper bounds on the number of embeddings of *generically minimally rigid* graphs for $d \geq 2$. As in [5], we apply an elimination process on a graphical structure to obtain upper bounds on the number of outdegree-constrained orientations for fixed d. In particular, we harness Maxwell's condition in order to restrict the degree of the eliminated vertices. We also treat vertices of certain degree profiles with a different approach from [5]. More precisely, we use a delicate method relating certain sequences with the elimination of path of vertices with these degree profiles. For the case of $d = 2$, our upper bound on the number of embeddings is of $\mathcal{O}(3.46^{|V|})$, while for Geiringer graphs the new upper bound is $\mathcal{O}(6.32^{|V|})$. Finally, we prove that there are graphs that can have outdegree-constrained orientations approaching our new upper bound in the case of $d = 2$.

Organization: We organize the thesis as follows. In Chapter 2, initially we introduce some basic concepts presented previously in [3, 5]. Subsequently, we provide some definitions and technical lemmas that will be used later for the elimination of vertices with certain degree profiles. In Chapter 3, we give detailed description of our elimination process in the case of dimension 2, establishing the new upper bound. This case shall serve as basis for some induction hypotheses in higher dimensions. In Chapter 4, we provide new upper bounds for $d \geq 3$ generalizing the results for Laman graphs. Then, in Chapter 5 we give examples of Laman graphs that have the biggest number of orientations among the cases we computed. These results give a higher significance on the tightness of our result. Finally, Chapter 6 we conclude and present some ideas that could extend the present research.

CHAPTER 2

PRELIMINARIES

In this chapter, we present some basic principles of graph rigidity. In addition to that, we state and prove several lemmata that will be useful in improving the new upper bound formula.

2.1 Hennenberg steps

In this section we present a method to construct *minimally rigid* graphs in dimension d, while starting from a complete graph K_d. The Hennenberg steps construct a superset of the set of minimally rigid graphs in dimension d. For $d = 2$, there are two such steps (see Figure 2.1); the Hennenberg type 1 step (H1) and Hennenberg type 2 step (H2). The H1 step, adds a vertex to the graph, and connects it, via edge, with two other vertices of the graph. The H2 step requires a subset of 3 vertices, say A, with at least one edge between them. Then, one of these edges is removed, and a new vertex is added to the set of vertices, along with 3 new edges, joining the new vertex with A. This operation can be seen as splitting an edge by adding a vertex between its two endpoints, which consists of the new vertex, and connecting it with another vertex of the graph.

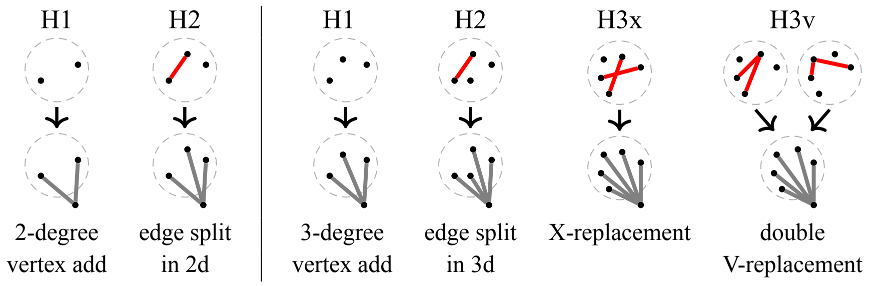

Figure 2.1: *Excerpt from [3]. Henneberg steps for Laman and Geiringer graphs.*

These two steps, H1 and H2, construct all the minimally rigid graphs in $d = 2$, that is the Laman graphs. In higher dimensions, there is a generalization of the Hennenberg steps. In dimension d, H1 step introduces a vertex and d edges that connects it with the graph, while H2 step, splits an edge, and connects the new vertex with $d-1$ other vertices. These steps, always preserve minimal rigidity when applied. However, these steps are not enough to construct all minimally rigid graphs in all dimensions, i.e. even in space we require another step, Hennenberg type 3 step (H3), to construct a superset of the corresponding minimally rigid graphs. In this new step, 2 edges are deleted and the new vertex is connected with the endpoints of the deleted edges and 1 additional vertex of the graph (see Figure 2.1 for the variations of this step). There is a conjecture that states the following [20].

Conjecture 2.1. The H1, H2, and H3 steps fully characterize the minimally rigid graphs in $\mathbb{R}^3$.

2.2 Algebraic Formulation

Let $G=(V,E)$ be a graph and $p=\{p_1,p_2,...,p_{|V|}\}\in\mathbb{R}^{d\cdot|V|}$ be an embedding of G in $\mathbb{R}^d$. Every such embedding induces a set of edge lengths $\lambda=(||p_v-p_u||)_{\{v,u\}\in E}$, where $||\cdot||$ denotes the euclidean norm, that must coincide with the given edge lengths, for the embedding to be valid. Given a set of edge lengths λ, the number of embeddings (or simply *embedding number*) are equal to the number of real solutions of an algebraic system that captures the edge lengths. A simple formulation of such a system is the following

$$\lambda_{v,u}^2=\sum_{j=1}^{d}(x_{v,j}-x_{u,j})^2\,,\quad \forall\{v,u\}\in E \tag{2.1}$$

where $x_{v,j}$ represents the i-th coordinate of vertex v. As it was mentioned before, the complex solutions of such systems extend the notion of the embedding in the complex space. In the sequel, as an algebraic system that captures the edge lengths, we will use the following formulation, that was also used in [9, 19], and is called *sphere equations* [2].

Definition 2.2 ([2])**.** Let $G=(V,E)$ be a graph. We denote by $\boldsymbol{\lambda}$ the lengths of the edges on G and by $\tilde{X}_u=\{x_{u,1},...,x_{u,d}\}$ the d variables that correspond to the coordinates of a vertex u. The following system of equations gives the embedding number for G:

$$\begin{gathered}||\tilde{X}_u||^2=s_u,\quad \forall u\in V\\ s_u+s_v-2\langle\tilde{X}_u,\tilde{X}_v\rangle=\lambda_{u,v}^2,\quad \forall(u,v)\in E\setminus E(K_d)\end{gathered}$$

where $\langle\tilde{X}_u,\tilde{X}_v\rangle$ is the Euclidean inner product. The first set of equations shall be called *magnitude equations*, while the second are the *edge* equations. Let $\tilde{X}_u$ contain an additional variable, i.e. $\tilde{X}_u=\{x_{u,1},...,x_{u,d},s_u\}$. By using $|\tilde{X}_u|=d+1$ coordinates and setting $s_u=1$, we can use this formulation to embed G on the unit d-dimensional sphere S^d.

Minimally rigid graphs in $\mathbb{R}^d$ are also minimally rigid in S^d, and vice versa [21]. Note that, other spaces or norms could be used to formulate the algebraic system of the embeddings [16, 21], but this would require different definitions and analysis in order to correspond to them.

To compute the *embedding number* of such a system, first we need to remove any rigid motions. To achieve that, we fix $\binom{d+1}{2}$ coordinates, hence the system becomes

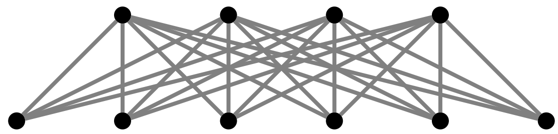

Figure 2.2: The complete bipartite graph $K_{6,4}$. It is minimally rigid in $\mathbb{R}^3$, but it does not contain a triangle.

0-dimensional. In the case of $d = 2$, we have to fix 3 coordinates, that is, both coordinates of a vertex and 1 coordinate of another vertex. This property has an extension in a general dimension d, in which, if a graph contains a complete subgraph on d vertices $K_d = \{v_1, ..., v_d\}$, then we can pick the coordinates of these vertices, so that the edge length constraints are satisfied. In the sequel, we refer to such a complete subgraph as *fixed*. It is reasonable to question, whether a minimally rigid graph in dimension d always contains a K_d. In the case of $d = 2$, this is true, but this is not true in higher dimensions. In particular, not all minimally rigid graphs in $\mathbb{R}^3$ contain triangle, i.e. bipartite $K_{6,4}$, which is a *Geiringer* graph that does not contain triangles (see Figure 2.2). This is obvious, since bipartite graphs do not contain cycles of odd length. Note that *Geiringer* graphs that do not contain triangles are very rare, and the 10-vertex $K_{6,4}$ is the first one.

2.3 Basic Lemmata and Notation

First, let us state the theorem that was proven in [3] that relates the number of embeddings of minimally rigid graphs in $\mathbb{R}^d$ with the number of d-orientations. Consequently, it gives us a combinatorial tool to calculate the multi-homogeneous Bézout. We have modified this theorem to fit the discussion on the possible absence of complete subgraphs with d vertices for a minimally rigid graph in $\mathbb{C}^d$ (see [3, 5] for more details on this subject and the underlying algebraic system).

Theorem 2.3 ([3]). Let $G = (V, E)$ be a minimally rigid graph in $\mathbb{C}^d$ that also contains a $K_{d'}$ as a subgraph, for some $d' \leq d$. The vertices of $K_{d'}$ are called *fixed*. If $d' < d$, then we introduce a set of *partially fixed* vertices, $V' = \{v_{d'+1}, ..., v_d\}$. Let $\mathcal{R}(G, K_{d'}, V')$ denote the number of orientations of $G' = (V, E \setminus E(K_{d'}))$, such that:

- the outdegree of the vertices of $K_{d'}$ is 0.
- if $d' < d$, the outdegree if the *partially fixed* vertices $v_{d'+1}, ..., v_d$ is $d - d' + 1, ..., d - 1$, respectively.
- the outdegree of every other vertex in G, that is every vertex in $V \setminus (V' \cup V(K_{d'}))$, is d.

The embedding number of G in $\mathbb{C}^d$ is bounded from above by

$$2^{|V|-d} \cdot \mathcal{R}(G, K_{d'}, V').$$

Corollary 2.4 ([3]). An H1 move always doubles the m-Bézout bound up to the same fixed $K_{d'}$. Moreover, if a graph can be constructed only with H1 moves, then the m-Bézout bound for this graph is exactly $2^{|V|-d}$.

Proof. Let $\mathcal{R}(G, K_{d'}, V')$ be the number of outdegree-constrained orientations for a graph $G(V, E)$ up to a given $K_{d'}$. This means that the m-Bézout bound is

$$mBe(G, K_{d'}) = 2^{|V|-d} \cdot \mathcal{R}(G, K_{d'}, V').$$

Now, let G^* be a graph obtained by an H1-move on the graph G. Since H1 adds a degree-d vertex to G, this means that there is only one way to reach outdegree d for the new vertex of G^*. So the outdegree-constrained orientations of G^* up to the same $K_{d'}$ are exactly $\mathcal{R}(G, K_{d'}, V')$ and

$$mBe(G^*, K_{d'}) = 2^{|V|+1-d} \cdot \mathcal{R}(G, K_{d'}, V') = 2 \cdot mBe(G, K_{d'}).$$

The second statement of this corollary can be proven by induction: Starting from K_d, only one orientation satisfies the requirements of Theorem 2.3 for each H1 move. So, the m-Bézout bound of a minimally graph constructed only by H1 moves is $2^{|V|-d}$. □

Remember that in $d = 2$, we always have a K_2, i.e. an edge. However, either V' is empty of non-empty, the asymptotic upper bound remains intact. So, now we are ready to focus on bounding the number of d-orientations, instead. In our approach, we apply an elimination process to the vertices of a graph, that at each step we increase a cost, that eventually expresses a bound on the number of d-orientations. To do that, we need to use a graphical structure that was also used in [5].

Definition 2.5. Let $J = (V_J, E_J, H_J)$ be a *pseudograph*, where V_J is the set of vertices, E_J is the set of edges, and H_J is an additional set of edges that only have one endpoint. The edges in H_J, have direction and they are directed outwards. To distinguish the two edge sets, we call the edges of E_J, *normal edges*, and the edges of H_J, *hanging edges*. The pair $J' = (V_J, E_J)$ is called *normal subgraph* of J.

As a consequence, every vertex v of a *pseudograph* has a degree profile (r, h), where r and h are equal to the number of *normal* and *hanging edges* incident to v, respectively. In [5], the authors use a different notation for the degree profile, that is (p, h), where h is the same as here, and p is equal to $r + h$. Here, since the focus is on vertices with the same *normal degree*, we change the notation.

Regarding our method, we start by removing a $K_{d'}$ out of G, and therefore we construct a *pseudograph* $J = (V_J, E_J, H_J)$, where $V_J = V(G) \setminus V(K_{d'})$, and $E_J = E(G) \setminus E_G(K_{d'})$. The hanging edges of J, consist of the edges that had the one endpoint in V_J and the other in $V(K_{d'})$, and the out-degree of the *partially fixed* vertices. If $V' \neq \emptyset$, then for every *partially fixed* vertex v that shall have outdegree $\hat{d}$, we shall consider $d - \hat{d}$ hanging edges. The number of d-orientations of J is equal to $\mathcal{R}(G, K_{d'}, V')$. These shall be called *valid d-orientations*, while every connected component of a pseudograph constructed as described above shall be a *connected d-pseudograph*. Let us remark that if a vertex v has degree profile (r, h) with $r + h < d$ or $h > d$, then it can have no valid d-orientation, since in the former case it can have outdegree strictly less than d, while in the latter case it has already outdegree greater than d.

Note that a minimally rigid graph in dimension d, is at least d-connected [17]. Hence, when we remove d-vertices out of it -in order to construct the *pseudograph*- we cannot ensure that the removal of every such set of vertices does not break the connectivity. However, in our analysis it suffices to bound the d-orientations of all connected

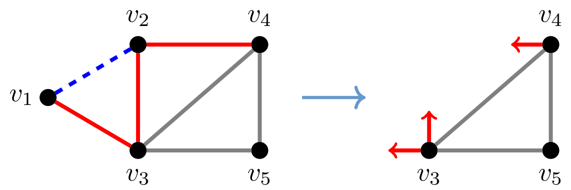

Figure 2.3: *An example of a Laman graph and a pseudograph constructed after the removal of a fixed K_2. (left) A Laman graph G, with fixed edge (v_1, v_2) (dashed blue). Since the fixed vertices have outdegree 0, their incident edges (red) are uniquely oriented. (right) The corresponding pseudograph. The red flexes represent the hanging edges. The degree profiles for vertices v_3, v_4, v_5 are respectively $(2, 2), (2, 1), (2, 0)$.*

$d-$pseudographs. In the following sections, we apply an elimination process to the vertices of such *pseudographs*. This process has the following stopping condition, already used in [5].

Lemma 2.6 ([5]). Let $J = (V_J, E_J, H_J)$ be a pseudograph such that $J' = (V_J, E_J)$ is a tree. Then

- the number of valid orientations for J is either 1 or 0,
- if G has a valid orientation, then $|H_J| = (d-1) \cdot |V_J| + 1$,

where d is the fixed outdegree required.

This count is derived from the relation $|E_J| = |V_J| + 1$ between the edges and the vertices of a tree and the fact that in order to have a valid $d-$orientation, it is needed a total of $d \cdot V_J$ edges and hanging edges. In other words, our goal is to eliminate every cycle from the normal subgraph J'. Notice that if we allowed the elimination of cut vertices, then the edge count for c connected trees would be $|E_J| = |V_J| + c$, so the relation between the hanging edges and the vertices would become $|H_J| = (d-1) \cdot |V_J| + c$. In order to restrict the parameters of the bound in our analysis only to total number of vertices and hanging edges, we prefer to keep the pseudograph connected throughout the elimination process.

Note that one could use a different stopping condition, namely a pseudograph that represents a graph that can be constructed by H1 steps. Corollary 2.4 tells us that such a graph can only have only one valid orientation [1], hence it can be considered as a stopping condition.

During the elimination process, the removal of a vertex v corresponds to the orientation of its incident edges. If an edge $e = (v, u)$ is outdirected from v, it is also removed in the next step of the elimination process. Otherwise, if e is directed inwards v, then it remains in the next step as a hanging edge incident only to u.

Each vertex removal has a *cost*, which expresses the number of valid orientation this vertex has, and it depends on the degree profile of this vertex, as well as the combinatorial properties of the pseudograph. A second quantity we use is the *hanging edges equilibrium* (H.E.E.). This gives a hint about how fast the elimination process may approach the tree condition.

[1] it would work even if we could find a condition on a graph that from that point henceforth the graph would have at most a constant number of orientations

Lemma 2.7. Let J be a pseudograph and v be one of its vertices with degree profile (r, h). The cost of the removal of a vertex v, expresses the quotient of the valid orientations of J over the maximum number of valid orientations of $J \setminus \{v\}$ and is equal to

$$\binom{r}{d-h} \tag{2.2}$$

while the H.E.E. is the difference between hanging edges in $J \setminus \{v\}$ and J

$$r - d \tag{2.3}$$

The *total cost* of the elimination process bounds the number of orientations. Notice that while the cost depends both on the normal and the hanging degree of a vertex, the H.E.E. depends only on the first one. This shall be used to group the elimination of vertices with different hanging degree, but the same normal one.

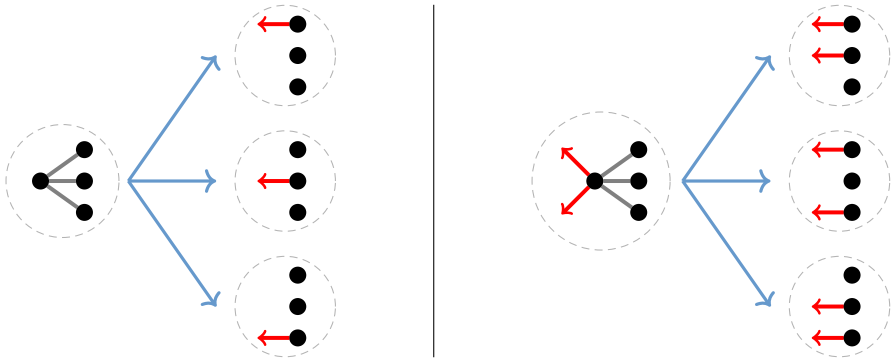

Figure 2.4: *The cost of a removal of a vertex is equal to the number of different valid ways of adding direction on its edges. (left) Removal of a* $(3, 0)$ *vertex in the case of* $d = 2$*. The cost in this case is* $\binom{3}{2} = 3$*. Two edges shall be deleted, therefore only one of its neighbours acquires a hanging edges. (right) Removal of a* $(3, 2)$ *vertex in the case of* $d = 3$*. This vertex already has 2 hanging edges, and needs one more to be saturated. Therefore the cost is* $\binom{3}{3-2} = 3$ *and every neighbour acquires a hanging edge in* 2 *scenarios.*

These general aspects of the elimination process were also used in [5]. Now we will present some new clues and concepts that will lead to the improved bounds.

First, remark that there are different scenarios for the distribution of hanging edges. Therefore, we will give an additional count that determines in how many cases a neighbour of the eliminated vertex acquires a new hanging edge or not.

Lemma 2.8. Let v be an eliminated vertex with degree profile (r, h) and u be one of its neighbours. Then there are exactly

$$\binom{r-1}{d-h} \text{ and } \binom{r-1}{d-h-1} \tag{2.4}$$

cases that u acquires or does not acquire a hanging edge respectively after the elimination of v.

Proof. Let us consider that u gets a hanging edge after the elimination of v. That means that the edge $e = (u, v)$ is directed towards v, so $d - h$ edges incident to v shall be directed outwards it. The available edges after the orientation of e are $r-1$, indicating that there are $\binom{r-1}{d-h}$ ways to orient them. Since the cost for the elimination of v is $\binom{r}{d-h}$, by Pascal's identity we derive the count for the case that u does not get a hanging edge. □

These quantities shall be used to determine the worst case scenarios for the vertices that are eventually eliminated with different degree profiles.

Now we show that there is always an elimination process that does not create more connected components of the d-pseudographs. Additionally, we impose certain restrictions on the normal degree of the removed vertex. For that reason, let us recall the definition of the block-cut tree.

Definition 2.9 ([12])**.** Let $G=(V,E)$ be a connected graph. There is a graph B_G, such that every vertex of B_G represents either a biconnected component in G, or an articulation point in G and its edges represent a biconnected component and an articulation point that belongs to that biconnected component. This graph is called the block-cut tree of G.

We may use the same definition for the block-cut tree (see Figure 2.5) of the normal subgraph of every pseudograph.

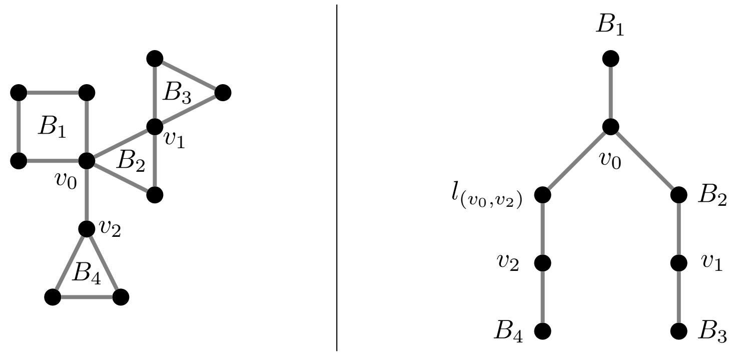

Figure 2.5: *(left) An example graph.* B_1*,* B_2*,* B_3 *and* B_4 *are the biconnected components. (right) The block-cut tree of the example graph.*

The following lemma uses Maxwell's condition to bound the normal degree of the eliminated vertices.

Lemma 2.10. Let $G=(V,E)$, $K_{d'}$ and V' inducing a pseudograph J as above. Then every connected component of $J=(V_J,E_J,H_J)$ has at least one non-cut vertex with normal degree smaller or equal than $2d-1$.

Proof. Due to the Maxwell condition, for every subgraph of $J'=(V_J,E_J)\subseteq G$, we get that $|E_J|\le d\cdot|V_J|-\binom{d+1}{2}$. Consider a leaf of the block-cut tree $B_{J'}$. We denote the biconnected component that corresponds to a leaf of $B_{J'}$ by L. The total normal degree of $J'[L]$ is at most $2d\cdot|V(J'[L])|-2\binom{d+1}{2}$. In it, there is at most one vertex which is a cut-vertex of J', because L is a biconnected component. The cut vertex has normal degree at least 2 in $J'[L]$. Assume that the smallest normal degree for a non-cut vertex in the leaf is $2d$. Then, it follows that the total normal degree is of $J'[L]$ at least

$2d \cdot |V(J[L])| - 2(d-1)$, violating Maxwell's condition. This leads to a contradiction, because Maxwell's condition shall be satisfied for $J'[L]$, so there are always vertices with normal degree smaller or equal to $2d-1$. □

Evidently, throughout the elimination process this property holds, since the normal subgraph of any pseudograph derived from an elimination step is always a subgraph of a minimally rigid graph. Notice that in [5], the bound on the valid $d-$orientations was related to all connected pseudographs, while this lemma restricts the analysis to pseudographs derived from minimally rigid graphs, i.e. $d-$pseudographs.

In our analysis, we need to distinguish two categories of elimination steps: *single vertex elimination step* and *path elimination step*. The latter is used to reduce the effect of vertices with degree profiles that would result to a bigger bound, if eliminated with the first method.

Path vertex elimination steps were also used in [5]. Here, we alter this method to group vertices with different hanging degree, but same normal degree. The following definition describes these paths.

Definition 2.11. Let $\mathcal{F}_{d,J}$ be a family of pseudographs which are subset of J. A graph $\mathcal{J}_d$ is contained into $\mathcal{F}_{d,J}$ if and only if $\mathcal{J}_d$ induces a maximal connected subgraph of J, such that the normal degrees in J of all vertices is the same and equal to $2d-1$ and has no cycles. Also $J - \mathcal{J}_d$ is connected.

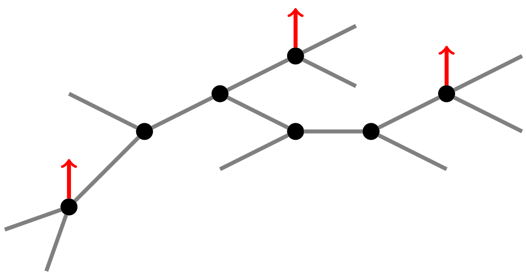

Figure 2.6: *An example subgraph $\mathcal{J}_2$ of a graph J that satisfies Definition 2.11. Notice that all the vertices have normal degree 3 on J, but all vertices but the first one are eliminated with normal degree 2.*

If $\mathcal{J}_d$ has more than one vertices, after the elimination of the first one with normal degree $2d-1$, all the other vertices in the path will be eliminated with normal degree $r = 2d-2$. In our analysis the first vertex is eliminated with a single vertex elimination step and the rest of the vertices with a path elimination step. Otherwise, if vertices with degree profile $(2(d-h), h)$ for $1 \leq h \leq d-1$ were always eliminated with a single vertex elimination step, then the analysis would lead to bigger bounds on d-orientations.

In order to use path elimination step we need to use a variant of the definition of the cost.

Definition 2.12. Let $\mathcal{J}_d = (v_0, v_1, \ldots, v_\ell)$ be a path of $\ell+1$ vertices, as in Definition 2.11, with $\ell \geq 1$. Let $C_d(\ell)$ be the *total cost* of removing these vertices in order. The **average cost** of removing the path without the first vertex v_0 is

$$\left(\frac{C_d(\ell)}{C_d(0)}\right)^{1/\ell}$$

where $C_d(0)$ is the cost of the removal of v_0.
Moreover if there is an effective upper bound $C_d^*(0)$ for $C_d(0)$, then the ratio

$$\left(\frac{C_d(\ell)}{C_d^*(0)}\right)^{1/\ell}$$

is the **eliminating average cost**.

Using these definitions, the total cost and the average cost can be computed by multiplying the cost of every elimination step, which can be either a single vertex elimination step, or a path elimination step. The eliminating average cost is used when we want to bound effectively the process by using an upper bound in the case of $C_d(0)$, instead of the exact cost. A necessary condition when using the average cost is that the H.E.E. is not altered by this change.

In the following sections, we show that the total cost of our paths follows the pattern of certain recursive sequences, in the worst case scenario. Here, we prove technical lemmas, that are used in order to bound eventually the cost of the path removal. First, we present a recursive formula for the total cost of the path, based on two other recursive functions.

Lemma 2.13. Let $\mathcal{B}_d(\ell)$ and $\mathcal{G}_d(\ell)$ be the following recursive functions:

$$\begin{aligned}\mathcal{B}_d(\ell+1) &= \frac{\alpha_d}{2}\cdot(\mathcal{B}_d(\ell)+\mathcal{G}_d(\ell))\\ \mathcal{G}_d(\ell+1) &= \frac{\alpha_d}{2}\cdot\mathcal{B}_d(\ell)+\left(\beta_d-\frac{\alpha_d}{2}\right)\cdot\mathcal{G}_d(\ell)\end{aligned}\tag{2.5}$$

where $\alpha_d=\binom{2d-2}{d-1}$ and $\beta_d=\binom{2d-2}{d}$, and $d\geq 2$. Given these functions we define the sequence

$$C_d(\ell)=\alpha_d\mathcal{B}_d(\ell)+\beta_d\,\mathcal{G}_d(\ell).\tag{2.6}$$

Then $C_d(\ell)$ is defined recursively for $l\geq 1$ by:

$$C_d(\ell+1)=\beta_d\cdot C_d(\ell)+\frac{\alpha_d(\alpha_d-\beta_d)}{2}\cdot C_d(\ell-1)\tag{2.7}$$

Proof.

$$\begin{aligned}C_d(\ell+1) &= \alpha_d\mathcal{B}_d(\ell+1)+\beta_d\,\mathcal{G}_d(\ell+1)\\ &= \frac{\alpha_d^2}{2}(\mathcal{B}_d(\ell)+\mathcal{G}_d(\ell))+\frac{\alpha_d\beta_d}{2}\mathcal{B}_d(\ell)+\beta_d(\beta_d-\frac{\alpha_d}{2})\mathcal{G}_d(\ell)\\ &= \frac{\alpha_d^2}{2}(\alpha_d\mathcal{B}_d(\ell-1)+\beta_d\mathcal{G}_d(\ell-1))+\frac{\alpha_d\beta_d}{2}(\mathcal{B}_d(\ell)-\mathcal{G}_d(\ell))\\ &\quad+\beta_d(C_d(\ell)-\alpha_d\mathcal{B}_d(\ell))\\ &= \frac{\alpha_d^2}{2}C_d(\ell-1)+\beta_d\mathcal{C}_d(\ell)-\frac{\alpha_d\beta_d}{2}(\mathcal{B}_d(\ell)+\mathcal{G}_d(\ell))\\ &= \beta_d\cdot C_d(\ell)+\frac{\alpha_d(\alpha_d-\beta_d)}{2}\cdot C_d(\ell-1)\end{aligned}$$

□

Notice that by the definition of the sequences, we have that $\mathcal{B}_d(\ell)>\mathcal{G}_d(\ell)$, since $\alpha_d>\beta_d$, but for the initial condition $\mathcal{B}_d(0),\mathcal{G}_d(0)$. In the sequel we set $\mathcal{B}_d(0)=\mathcal{G}_d(0)=1$, except for the mixed path case treated in Lemma 4.4.

Now we are ready to bound the ratio of two consecutive terms of the sequence $C_d(\ell)$. This is used to bound the cost of vertices eliminated with a path elimination step.

Lemma 2.14. For all $\ell \geq 0$ and $d \geq 2$ the ratio $\dfrac{C_d(\ell+1)}{C_d(\ell)}$ is strictly bounded from above by

$$\mathcal{D}(d) = \frac{\alpha_d^2 + \beta_d^2}{\alpha_d + \beta_d},$$

if the initial cost $C_d(0) = \alpha_d + \beta_d = \binom{2d-1}{d}$.

Proof. First, we will prove that it holds for $\ell \geq 4$, and then we will prove it for the 4 starting cases. The d subscripts are omitted in this proof, because they are not altered.

$$\begin{aligned}
&\frac{C(\ell+1)}{C(\ell)} \leq \frac{\alpha^2+\beta^2}{\alpha+\beta} && \iff \\
&(\alpha+\beta)\cdot C(\ell+1) \leq (\alpha^2+\beta^2)\cdot C(\ell) && \iff \\
&(\alpha+\beta)\cdot C(\ell-1) \leq 2\cdot C(\ell) && \iff && (2.8)\\
&(\alpha+\beta)\cdot C(\ell-1) \leq 2\beta\cdot C(\ell-1) + \alpha(\alpha-\beta)\cdot C(\ell-2) && \iff \\
&C(\ell-1) \leq \alpha\cdot C(\ell-2) && \iff && (2.9)\\
&\beta\cdot C(\ell-2) + \frac{\alpha(\alpha-\beta)}{2}\cdot C(\ell-3) \leq \alpha\cdot C(\ell-2) && \iff \\
&\alpha\cdot C(\ell-3) \leq 2\cdot C(\ell-2) && \iff && (2.10)\\
&\alpha\cdot C(\ell-3) \leq 2\beta\cdot C(\ell-3) + \alpha(\alpha-\beta)C(\ell-4) && \iff \\
&(\alpha-2\beta)\cdot C(\ell-3) \leq \alpha(\alpha-\beta)\cdot C(\ell-4)
\end{aligned}$$

which is true since $\alpha \leq 2\beta$ for $d \geq 2$ and the other factors are positive.

Now, we should check if it holds for the remaining cases, i.e. for $C(1)/C(0)$, $C(2)/C(1)$, $C(3)/C(2)$, and $C(4)/C(3)$.

For $C(1)/C(0)$, we simply use the definition $C(\ell+1) = \alpha\mathcal{B}(\ell+1) + \beta\mathcal{G}(\ell+1)$, for $\ell = 0$ and we find the values of $\mathcal{B}(1)$ and $\mathcal{G}(1)$ by using their definition. Recall that $\mathcal{B}(0) = \mathcal{G}(0) = 1$. Hence, we have the following equality:

$$\frac{C(1)}{C(0)} = \frac{\alpha^2+\beta^2}{\alpha+\beta}.$$

For the case of $C(2)/C(1)$, we stop at the inequality 2.8 above, for $\ell = 1$, and we have

$$\begin{aligned}
(\alpha+\beta)\cdot C(0) &\leq 2\cdot C(1)\\
(\alpha+\beta)^2 &\leq 2\cdot(\alpha^2+\beta^2)\\
(\alpha-\beta)^2 &\geq 0
\end{aligned}$$

which is true.

For the case $C(3)/C(2)$, we stop at the inequality 2.9, for $\ell = 2$:

$$C(1) \leq \alpha \cdot C(0)$$
$$\alpha^2 + \beta^2 \leq \alpha^2 + \alpha\beta$$

which is true, since $\alpha \geq \beta$.

For the last case, consider inequality 2.10 at $\ell = 3$. We have that

$$\alpha \cdot C(0) \leq 2 \cdot C(1)$$
$$\alpha\beta \leq \alpha^2 + 2\beta^2$$

which is clearly true. □

Lemma 2.14 clearly shows that the following inequality holds for the average cost of every sequence defined as in Lemma 2.13.

$$\left(\frac{C_d(\ell)}{C_d(0)}\right)^{1/\ell} \leq \left(\frac{C_d(1)}{C_d(0)}\right)^{1/\ell}.$$

CHAPTER 3

LAMAN GRAPHS

In this chapter, we develop a method that improves the existing upper bounds on the embedding number of Laman graphs. The analysis for this dimension is simpler than the ones regarding higher dimension and serves as base case for higher dimensions. Our method relies on Theorem 2.3, which relates the bound on the embedding number with the outdegree-constrained orientations. In this section we can remove subscripts referring to the embedding space from $\mathcal{F}_{d,J}, \mathcal{J}_d, \mathcal{C}_d, \mathcal{B}_d, \mathcal{G}_d, \alpha_d, \beta_d$.

We use an elimination process similar to [5] in order to improve the upper bound. We remind that the tree condition (see Lemma 2.6) signifies the termination of the process. Given a pseudograph $J = (V_J, E_J, H_J)$ and setting $n = |V_J|$ and $k = |H_J|$, we have that this condition is satisfied if $k = n + 1$.

One of the main differences between the elimination method described here and the one in [5], is the restriction on the normal degree of the eliminated vertices. Specializing Lemma 2.10 to the case of $d = 2$, we have that connected 2-pseudographs derived from the deletion of a fixed edge in a Laman graph have always a non-cut vertex with normal degree less or equal to 3. This also happens for all connected pseudographs that evolve through the elimination process, signifying that these that have valid 2-orientations may have vertices with the following vertex profiles, cost and H.E.E. (see Equation 2.2):

- Vertices with normal degree 1 have H.E.E. = -1: (1, 2), (1, 1), with cost=1.
- Vertices with normal degree 2 have H.E.E.= 0: (2, 2), (2, 0), with cost = 1, and (2, 1), with cost = 2.
- Vertices with normal degree 3 have H.E.E.= 1: (3, 2), with cost = 1, and (3, 0), (3, 1), with cost = 3.

The vertices that have cost=1 will be called *trivial vertices* in the sequel, since their removal does not increase the total cost of the elimination process. Now we will describe the elimination process and the different cases treated. All vertices with normal degree 3 are eliminated with a single vertex elimination step, their cost is bounded by 3 and generate 1 hanging edge. For the vertices with normal degree 2, we consider a dichotomy described in the following definition.

Definition 3.1. We consider a pseudograph J and an elimination process bounding its cost. The *non-composite vertices* with normal degree 2 are the eliminated vertices that

- had already normal degree 2 in J.
- have normal degree 2 and they were generated by the removal of another non-composite trivial vertex with normal degree 2 or by the removal of a vertex with normal degree 1.

All the other vertices eliminated with normal degree 2 are called *composite*.

Notice that since non-composite $(2,1)$ vertices have one hanging edge and the H.E.E. of trivial vertices that may generate them is ≤ 0, during the elimination process of a pseudograph with k hanging edges. Thus, there can be eliminated at most k of these vertices.

The composite vertices can be grouped in order to fit Definition 2.11 and subsequently the worst case scenario for their elimination follows Lemma 2.13 bounding the average cost from the quantity indicated in Lemma 2.14. The dichotomy described and the grouping are essential, because if single vertex elimination was considered for all $(2,1)$ vertices, then the bound would be higher. This is the delicate part of our analysis.

The following lemma shows that we can consider only composite vertices $(2,1)$ in $\mathcal{J}$ for our elimination process.

Lemma 3.2. There is always an elimination process such that all composite non-cut vertices are created after the elimination of a vertex in $\mathcal{F}_J$.

Proof. The only way to create a vertex with normal degree 2 is by eliminating the neighbour of a vertex with normal degree 3. By Definition 2.11, if the eliminated vertex has also degree 3, then it belongs to $\mathcal{F}_J$, so our case holds.

If at a certain instance of the elimination $\mathcal{F}_J = \emptyset$, then either there are no vertices with normal degree 3, or all such vertices are cut vertices. In the first case, there is nothing to prove. In the second case, we can continue the elimination process eliminating a vertex that lies in the leaf of the block cut tree. By Lemma 2.10 there is always a non-cut vertex u in this biconnected component, with normal degree smaller than 3. If the cut vertex v has normal degree 3 and u has normal degree 1, then after its elimination v is a non-composite vertex. If u has normal degree 2, then v remains a cut vertex and cannot be eliminated before a further drop of degree in one of the next elimination steps. □

It is clear from Definition 2.11, that all vertices of $\mathcal{J}$ but the first one are eliminated with normal degree 2, since when one vertex is eliminated, then the normal degree for all its neighbours drops by 1 in the resulting pseudograph. The following corollary shows how we can bound the average cost of such paths in the case of Laman graphs.

Lemma 3.3. The eliminating average cost for the elimination of composite vertices can be set as less or equal to $5/3$. This is a specialization of Lemma 2.14 for $d = 2$.

Proof. We show that the worst case scenario for the total and the average cost is covered by Equation 2.7 that results from the recursive Equations 3.1 (see Lemma 2.8 for details). It is obvious that in the ℓ-th move the cost is exactly $C(\ell) = 2\mathcal{B}(\ell) + \mathcal{G}(\ell)$, where $\mathcal{B}(\ell), \mathcal{G}(\ell)$ denote the cardinality of vertices with degree profile $(2,1)$ and $(2,0)$ or $(2,2)$ (which are trivial vertices) respectively. This definition for $C(\ell)$ is a specialization of Equation 2.6 in the case of $d = 2$. Lemma 2.8 gives the scenarios for the distribution of the hanging edge. Thus, the elimination of a $(2,1)$ vertex results to two different scenarios, indicating that the neighbour in the path becomes a $(2,1)$ vertex in half of the cases, and a trivial one in the other cases (see Figures 3.1, 3.2). On the other hand for the neighbour of a trivial vertex there is only one scenario, it will be either $(2, 1)$ (see Figure 3.1) or again trivial (see Figure 3.2). Let us denote by $\mathcal{G}^*(\ell)$ the number of trivial vertices that create $(2, 1)$ vertices and $\mathcal{G}'(m)$ the number of trivial vertices that create other trivial vertices. This implies that $\mathcal{G}(\ell) = \mathcal{G}^*(\ell) + \mathcal{G}'(\ell)$. So, the cardinalities for the next vertex in $\mathcal{J}$ (if such exists) are:

$$\mathcal{B}(\ell + 1) = \mathcal{B}(\ell) + \mathcal{G}^*(\ell)$$
$$\mathcal{G}(\ell + 1) = \mathcal{B}(\ell) + \mathcal{G}'(\ell)$$

leading to the following relation for the total cost of a path in the $(\ell + 1)$-th step:

$$\begin{aligned} C(\ell + 1) &= 2\mathcal{B}(\ell + 1) + \mathcal{G}(\ell + 1) \\ &= 2\mathcal{B}(\ell) + 2\mathcal{G}^*(\ell) + \mathcal{B}(\ell) + \mathcal{G}'(\ell) \\ &= 2\mathcal{B}(\ell) + (\mathcal{G}^*(\ell) + \mathcal{G}'(\ell)) + \mathcal{B}(\ell) + \mathcal{G}^*(\ell) \\ &\leq C(\ell) + \mathcal{B}(\ell) + \mathcal{G}(\ell) \\ &= C(\ell) + 2\mathcal{B}(\ell - 1) + \mathcal{G}^*(\ell - 1) + \mathcal{G}'(\ell - 1) \\ &= C(\ell) + C(\ell - 1) \end{aligned} \tag{3.1}$$

The last quantity proves our point[1].

We need to specify the different initial conditions of the path in order to prove that the total cost of the path permits to use $\mathcal{D}(2) = 5/3$ as an upper bound for the average cost. By Lemma 3.2, we consider only the elimination for paths of vertices in $\mathcal{J}$, so the initial vertices can have only normal degree 3. If v_0 is a $(3, 0)$ or a $(3, 1)$ vertex and has cost 3, then the sequence $C(\ell)$ in the worst case scenario is exactly the one of Lemma 2.13 for $d = 2$.

If v_0 is a $(3, 2)$ vertex and v_1 is eliminated as a $(2, 1)$ vertex, then the ratio $C(1)/C(0) = 2 > 5/3$. We overcome this situation by making use of the eliminating average cost setting that $C^*(0) = 3$, while $C(1)$ is not altered. This change cannot surpass the number of orientations in our analysis, since the total cost of the path is not altered. Furthermore, for for single vertex elimination in the case of $r = 3$ we have already considered a bound for the cost of vertices with such normal degree, as mentioned before, which is 3. Since $C(1)$ is smaller than the respective value of the sequence in Lemma 2.13, the next terms will be also smaller, so the eliminating average cost for all vertices but the first one is strictly bounded by $5/3$. □

Now we are ready to bound from above the number of valid 2-orientation.

Theorem 3.4. The total number of 2-orientations for a connected 2-pseudograph with n vertices and k hanging edges derived by will be at most

$$3^{(n+1)/2} \cdot (2/3)^k$$

Proof. We consider that throughout the elimination process there have been removed t vertices with normal degree 3, m non-composite $(2, 1)$ vertices, ℓ vertices with normal degree 2 in paths $\mathcal{J}$, s_2 trivial non-composite vertices and s_1 vertices with normal degree 1. Recall that the elimination process stops when the tree condition is satisfied. Neglecting trivial vertices, the total cost is bounded by

[1]Notice that the worst case scenario for the cost sequence in dimension 2 has the same recursive definition as the Fibonacci sequence.

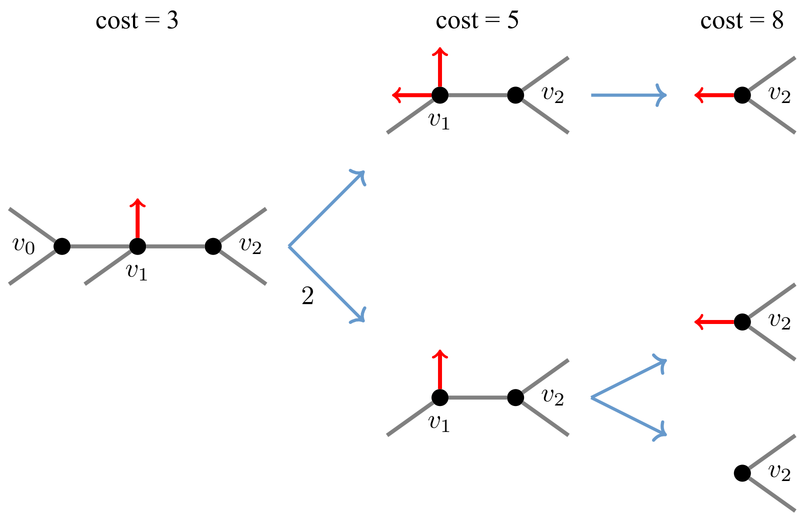

Figure 3.1: *An example of a path elimination step (left to right). Weights on blue flexes show that a result is produced multiple times. Above each step, there is the cost of the corresponding removal. (left) The first vertex v_0 has degree profile $(3, 0)$ and is eliminated with a single vertex elimination step. (middle) There are 3 different scenarios for the distribution of hanging edges after the elimination of v_0. In 2 of them v_1 becomes a $(2, 1)$ vertex, while in the other case it becomes a $(2, 2)$ vertex. The total cost for the removal of v_1 adding all cases is $2 \cdot 2 + 1 = 5$. (right) The elimination of vertex v_2 follows the same principle. The average cost for the elimination of v_1 and v_2 is $(8/3)^{1/2} < 5/3$. Notice that if we did not apply the path elimination step, we should have eliminated vertices v_1 and v_2 with cost 2 (which is the biggest) and the total cost would be 12.*

$$3^t \cdot 2^m \cdot (5/3)^\ell \tag{3.2}$$

Now we will use the tree condition in order to find a bound up to n and k. If the final number of vertices and hanging edges in the elimination process are n' and k' respectively then we have the following equations:

$$\begin{aligned} n' &= n - t - m - \ell - s_1 - s_2 \\ k' &= k + t - s_1 \end{aligned}$$

Since $n' + 1 = k'$, we conclude that

$$t \leq \frac{n - k - m - \ell + 1}{2}.$$

This results to

$$\begin{aligned} 3^t \cdot 2^m \cdot (5/3)^\ell &\leq 3^{n/2} \cdot 3^{-k/2} \cdot 3^{-m/2} \cdot 3^{-\ell/2} \cdot 3^{1/2} \cdot 2^m \cdot \left(\frac{5}{3}\right)^\ell \\ &= 3^{n/2} \cdot \left(\frac{2}{3}\right)^k \cdot \left(\frac{5}{3^{3/2}}\right)^\ell \cdot 3^{1/2} \\ &\leq 3^{n/2} \cdot \left(\frac{2}{3}\right)^k \cdot 3^{1/2} \end{aligned}$$

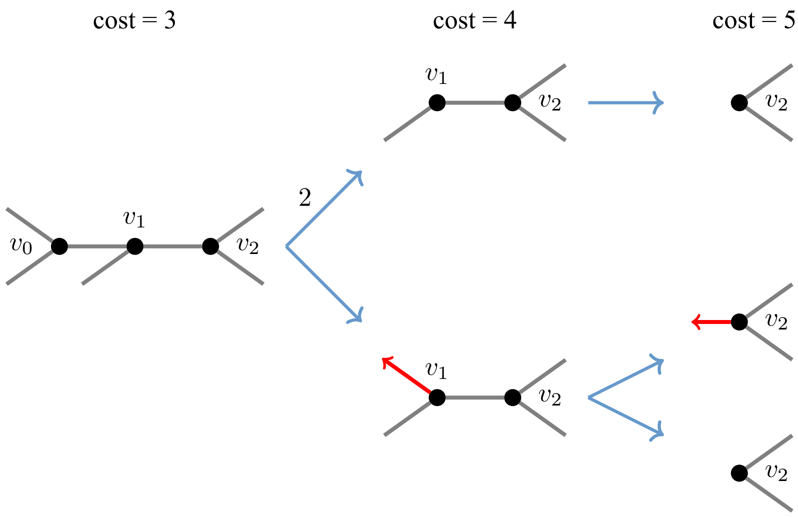

Figure 3.2: *Another example of a $\mathcal{J} \in \mathcal{F}_J$ that induces a path in J. The example is similar to Figure 3.1, with the only difference that v_1 is a $(3, 0)$ vertex now and the total cost is lower. In this case, notice that the removal of v_0, results to 2 trivial and 1 non-trivial cases for v_1.*

since m is at most k. □

Subsequently we have that given a pseudograph J derived from a Laman graph with c connected components, then its number of valid 2-orientations is bounded by:

$$3^{(n+c)/2} \cdot \left(\frac{2}{3}\right)^k. \tag{3.3}$$

Nevertheless, we prove that an exact bound on the embedding number of Laman graphs can be derived considering a fixed edge, whose removal does not create multiple components. Recall from graph theory that given a connected graph $G = (V, E)$, a subset of vertices $S \subseteq V$ is called *vertex separator* is its removal breaks the connectivity.

Lemma 3.5. Let G be a Laman graph, then $\exists e = (u, v) \in E(G)$ such that $G' = (G \setminus \{u\}) \setminus \{v\}$ is connected.

Proof. Since G is a Laman graph, it is at least 2-connected [17]. If the minimum size separator contains at least 3 vertices, then the lemma is proven. In the other case, we denote a 2-vertex separator with S_1. If we remove S_1 from G, then we get two components G_1, G_1'.

If one of the two components, i.e. G_1, does not contain a separator, then S_1 is called *extreme*. This means that either $G[V(G_1)]$ contains edges, but the deletion of any 2 vertices does not break the connectivity, or there are no edges in $G[V(G_1)]$. In the latter case, every edge that is incident to a vertex in G_1, has its other endpoint in S_1. If S_1 is not an extreme separator, we repeat the process in G_1 without loss of generality, setting a new partition in G_2, S_2, G_2' as before. We end the process when a separator in one of the two components is extreme.

Let us denote the two components and the separator in the end of this process with G_S, G'_S and S respectively. We consider G_S to be the component with no separator. If there is an edge in one of the components, then trivially the deletion of its endpoints does not break the connectivity.

If there is no edge in G_S, then let u, u' be the vertices in S and v be a vertex in G_S. Since G is 2-connected, then v has degree at less 2, so there are edges (u, v) and (u', v). Since there are no edges in G_S and both u and u' connect with both G_S and G'_S, the removal of v and one of these 2 cannot break the connectivity. □

Theorem 3.6. Let $G = (V, E)$ be a Laman graph. The embedding number of G in $\mathbb{C}^2$ and S^2 is bounded from above by

$$\frac{16}{3^{7/2}} \cdot \left(2 \cdot 3^{1/2}\right)^{|V|-2}.$$

The asymptotic order of this bound is $\mathcal{O}(3.46^{|V|})$.

Proof. In [3] it is proven that the bound of the embedding number for a Laman graph G with a 2-valent vertex v is the same with the number of orientations of the graph $G \setminus \{v\}$, so by Theorem 2.3 the number of orientations is the same. This means that for the general bound, we may consider only Laman graphs with minimum degree 3. Since the number of hanging edges k is equal to the number edges incident to the fixed vertices, but for the fixed vertex, we have that $k \geq 4$. Also, Lemma 3.5 indicates that there is always a fixed edge whose removal does not break connectivity.

By setting $k = 4$ and $c = 1$ in the upper bound on the 2-orientations given in Equation 3.4, and combining it with Theorem 2.3 we derive the new upper bound. □

CHAPTER 4

HIGHER DIMENSIONS

In this chapter we improve the bounds for all $d \geq 3$. The elimination method here is analogous with $d = 2$. The main difference is that paths with multiple normal degree profiles shall be considered. Remark also that the case of Laman graphs is used as base case for some of our proofs. The asymptotic bound derived in this section is given by the following Theorem, that extends Theorem 3.6.

Theorem 4.1. Let $G = (V, E)$ be a *minimally rigid* graph in dimension $d \geq 2$. The embedding number of G is bounded from above by

$$\mathcal{O}\left(\left(2 \cdot \binom{2d-1}{d}^{1/2}\right)^{|V|}\right).$$

Let $\mathcal{C}_d(r, h)$ denote the cost of the removal of a vertex with r normal edges and h hanging edges in dimension d. The following quantity

$$\mathcal{C}_d(r, h)^{\frac{d-1}{r-1}} \tag{4.1}$$

is the *asymptotic effect* for the elimination of a vertex with degree profile (r, h), for $r \geq 2$. Notice that the exponent in this quantity is not affected by the hanging degree of the vertices, so it is the same for all vertices with the same normal degree. We also consider that the asymptotic effect of vertices with normal degree 1 is trivially 1. Let us also remind that there is no reason to examine vertices such that $r + h < d$ or $h > d$, since they have no valid d-orientations.

The first step leading to the bound of d-orientations is to derive the maximal of the asymptotic effect for certain cases of degree profiles. More precisely we prove that the asymptotic effect of $(2d-1, 0)$ vertices is bigger or equal to the asymptotic effect of all other vertices examined in the following lemma. Thus, $\mathcal{C}_d(2d-1, 0)^{\frac{1}{2}}$ shall be called *target bound* in the sequel.

Lemma 4.2. The asymptotic effect is maximized over all $1 \leq r \leq 2d-1$ with $0 \leq h \leq d$ for $(r, h) = (2d-1, 0)$, but for the cases $(r, h) = (2(d-h), h)$ with $1 \leq h \leq d-1$.

Proof. The case of $r = 1$ is trivial by definition.

For the non-trivial cases, we prove initially that $\mathcal{C}_d(r,0)^{\frac{d-1}{r-1}}$ is monotonically increasing for all integers in $r \in [d, 2d-1]$. This corresponds to the asymptotic effect of vertices with no hanging edges. We are interested only in this interval since by Lemma 2.10 there is an elimination process such that the maximum normal degree is $2d-1$ and $r+h \geq d$. Observe that the term $d-1$ is constant, so it suffices to prove that the ratio

$$U_d(r) = \frac{\mathcal{C}_d(r+1,0)}{\mathcal{C}_d(r,0)} = \frac{(r+1)^{r-1}}{(r+1-d)^{r-1} \cdot \binom{r}{d}}$$

is bigger than 1 in this interval.

If we take the ratio $\dfrac{U_d(r+1)}{U_d(r)}$ we conclude that

$$\frac{U_d(r+1)}{U_d(r)} = \left(\frac{(r+2)(r+1-d)}{(r+1)(r+2-d)}\right)^r$$

is always smaller than 1 $\forall d \geq 2$, so $U(r)$ decreases and its minimum in $[d, 2d-1]$ is

$$U^*(d) = U_d(2d-1) = \frac{4^{2d-2}}{\binom{2d-1}{d}}.$$

For $d = 2$ we have that $U^*(d) = 4/3 > 1$ and it can be easily checked that $\dfrac{U^*(d+1)}{U^*(d)} > 1$ proving that $\forall d \geq 2$

$$U_d(2d-1) > 1.$$

This implies that, we have that $\mathcal{C}_d(r+1,0)^{1/r} > \mathcal{C}_d(r,0)^{1/(r-1)}$ for every $r \in [d, 2d-1]$, concluding our claim.

The cases for $1 \leq h \leq d$ and $d-h \leq r \leq 2 \cdot (d-h) - 1$ can be related with the bounds on the orientations in lower dimensions. Thus, we prove by induction that if the maximum for the asymptotic effect holds for $d-h$ it also holds for these cases for d. We just proved that

$$\mathcal{C}_{d-h}(r,0)^{(d-h-1)/(r-1)} \leq \mathcal{C}_{d-h}(2(d-h)-1,0)^{1/2}$$

holds if we consider $d^* = d-h$.

So we need to prove the last part of the following inequality

$$\begin{aligned} \mathcal{C}_{d-h}(r,0)^{(d-1)/(r-1)} &\leq \mathcal{C}_{d-h}(2(d-h)-1,0)^{(d-1)/(2(d-h)-2)} \\ &= \mathcal{C}_d(2(d-h)-1,h)^{(d-1)/(2(d-h)-2)} \\ &\leq \mathcal{C}_d(2d-1,0)^{1/2}. \end{aligned} \tag{4.2}$$

This will be done by demonstrating that the ratio

$$\frac{\mathcal{C}_d(2(d-h)-1,h)^{2(d-h-1)-2}}{\mathcal{C}_d(2(d-h-1)-1,h+1)^{2(d-h)-2}}$$

is always bigger than 1 for $0 \leq h \leq d$. In other words, the shift $h \to h+1$ reduces the asymptotic effect. Considering $d^* = d-h$ as above, this shift turns to $d^* \to d^*-1$

and the ratio becomes

$$W(d^*) = \frac{\mathcal{C}_{d^*}(2d^*-1,0)^{2d^*-4}}{\mathcal{C}_{d^*}(2d^*-3,1)^{2d^*-2}} = \left(\frac{4d^*-2}{d^*}\right)^{2d^*-4} \frac{1}{\binom{2d^*-3}{d^*-1}^2}$$

which is bigger than 1 for $d^* = 3$, since $10^2 > 3^4$.

Now the ratio

$$\begin{aligned} \frac{W(d^*)}{W(d^*-1)} &= \left(\frac{(4d^*-2)(d^*-1)}{(4d^*-6)d^*}\right)^{4d^*-6} \cdot \left(\frac{\binom{2d^*-3}{d^*-1}}{\binom{2d^*-5}{d^*-2}}\right)^2 \\ &= \left(1 + \frac{2}{4d^{*2} - 6d^*}\right)^{4d^*-6} \end{aligned}$$

is also bigger than 1, showing that $W(d^*)$ is increasing.

What remains is to deal with the case of $(2(d-h), h^*)$ vertices with $h^* \geq h+1$, as well as the case of vertices with $r \geq 2(d-h)+1$ and more or equal than h hanging edges. Notice that in the first case if the term $2(d-h)$ in

$$\binom{2(d-h)}{d-h^*},$$

if fixed, then this binomial coefficient decreases as $h^* \geq h$ increases. Since we have that $\mathcal{C}_d(2(d-h), h+1) = \mathcal{C}_d(2(d-h), h-1)$ and vertices $(2(d-h), h-1)$ have smaller asymptotic effect than $(2(d-h)+1, h-1)$ vertices as proven before, our hypothesis is valid.

The same comparison with $(2(d-h)+1, h-1)$ vertices can be done for $(2(d-h)+1, h^*)$ with $h^* \geq h$, since $\mathcal{C}_d(2(d-h)+1, h) = \mathcal{C}_d(2(d-h)+1, h-1)$ and all vertices with more hanging edges have smaller cost. Finally, we remark that in the previous 2 cases the cost function was maximized for vertices with h hanging edges, from the properties of binomial coefficients. Thus, if we want to examine the case of $r \geq 2(d-h)+2$, we can refer to the cases of vertices with less hanging edges that shall have bigger cost. Our base case now, are the $(2d-1, 1)$ vertices and $(2d-1, 2)$ ones. The first have asymptotic effect equal to the target bound, while the latter have strictly smaller, concluding our proof. □

The cases of vertices with degree profile $(2(d-h), h)$, cannot be included in this kind of analysis for all dimensions, since the ratio

$$\frac{C_d(2(d-h), h)^{2d-2}}{C_d(2d-1, 0)^{2(d-h)-1}}$$

is strictly bigger than 1. Notice that this case is treated in dimension 2 for $(2, 1)$ vertices. What's more, this condition is inherited for vertices with the same normal degree and increased hanging degree in bigger dimensions. For example in dimension 3, the vertices with higher asymptotic effect than the target bound are both the $(4, 1)$ vertices and the $(2, 2)$ vertices. The latter correspond to the $(2, 1)$ vertices in dimension 2, since they have the same cost.

We will treat these cases expanding the idea of grouping composite vertices presented in Section 3. Let us define the dichotomy between composite and non-composite vertices in general dimension. We remind that trivial vertices are the ones that have cost equal to 1.

Definition 4.3. Let J be a pseudograph, in which we apply an elimination process to bound the number of its $d-$orientations. The *non-composite vertices with normal degree* $2(d-h)$, for $1 \leq h \leq d-1$ are the vertices with degree profile $(2(d-h), h), (2(d-h), h+1), (2(d-h), h-1)$ such that

- they had exactly this degree in J.
- they have this degree profile and they were generated by the removal a trivial vertex with normal degree $r \leq d$.

All the other vertices eliminated with this degree profile are called *composite vertices with normal degree* $2(d-h)$.

This definition serves to group all composite vertices for different path elimination steps. The non-composite vertices with degree profile $(2(d-h), h+1), (2(d-h), h-1)$ have smaller asymptotic effect than the target bound, while the cardinality $(2(d-h), h)$ vertices is bounded by the number of initial hanging edges, since they can be generated only by a drop in H.E.E.

Analogously with Lemma 3.2, there is always an elimination process such that no composite non-cut vertex can be generated by vertices with degree profile other that the ones belonging in the previous definition. The following lemma bounds the average cost for these vertices.

Lemma 4.4. The removal of a composite path with normal degree $2(d-h)$ has eliminating average cost at most $\mathcal{D}(d-h+1)$.

Proof. First, we show that the cost function follows at worst case the recursion Equation 2.7. We will consider the case of a $\mathcal{J}_d$ path with more than one vertices, which can be generalized in all other cases. Let $\mathcal{B}_d(\ell)$ and $\mathcal{G}_d(\ell)$ denote the cardinality of a $(2d-2, 1)$ and $(2d-2, 0)$ or $(2d-2, 2)$ vertices respectively. Since vertex v_0 is eliminated with normal degree $2d-1$, so $\alpha_d+\beta_d = \mathcal{C}(2d-1, 0)$, we set $\mathcal{B}_d(0) = \mathcal{G}_d(0) = 1$ which satisfies the cost function and the count for the distribution of hanging edges (see Lemma 2.8). By the same Lemma, the elimination of a $(2d-2, 1)$ vertex gives a hanging edge to one of its neighbours in exactly $\binom{2d-3}{d-1}$ cases, while in the rest $\binom{2d-3}{d-2}$ cases this neighbour does not acquire any hanging edge. Both these quantities are equal to $\alpha_d/2$. Similarly in the cases of $(2d-2, 2)$ vertices, a neighbour acquires a hanging edge in $\beta_d - \alpha_d/2$ cases and does not get any in $\alpha_d/2$ cases, while for $(2d-2, 2)$ vertices this counts are reversed. This means that the worst case scenario for the cardinalities follows Equation 3.1, leading to the desired recursive function for the cost. This gives the upper bound $\mathcal{D}(d)$ for the average cost if the first vertex is eliminated as a vertex with degree profile $(2d-1, 0)$. If the first vertex has normal degree $(2d-1, h)$ with $h \geq 2$, we can make a similar modification as with $(3, 2)$ vertices in the case of Laman graphs (see Lemma 3.3) and use the eliminating average cost for the paths.

The case of the removal of a single path is proven, but definition 4.3 and the adjustment of Lemma 3.2 in dimensions $d \geq 3$ allow the generation of composite vertices in $\mathcal{J}_{d-h}$ with $h \geq 1$, after the removal of another path $\mathcal{J}_d$. So it remains to prove that the removal of all vertices in both paths does not violate the eliminating average cost in our analysis.

First we consider the case of the elimination of a $\mathcal{J}_d$ path followed by the elimination of a $\mathcal{J}_{d'}$ path, with $d > d'$. Notice that if the removal of both paths increased the connected components, the elimination would not be valid. Thus, we are allowed to remove the paths with any order.

The cost for elimination of the first path $\mathcal{J}_d$ follows the count set above. Let now $\mathcal{B}_{d'}(0), \mathcal{G}_{d'}(0)$ denote the cardinalities of vertices with normal degree $2d-2$ in the final step of the $\mathcal{J}_d$ removal, if $\mathcal{J}_d$ has more than 1 vertices, otherwise $\mathcal{B}_{d'}(0) = \mathcal{G}_{d'}(0) = 1$. $\mathcal{B}_{d'}(1)$ and $\mathcal{G}_{d'}(1)$ are the cardinalities of vertices with normal degree $2d' - 2$ in the first step of elimination of $\mathcal{J}_{d'}$. We also denote the cost of the paths $\mathcal{C}_{d'}(0)$ and $\mathcal{C}_{d'}(1)$ respectively. The worst case scenario for the cost $\mathcal{C}_{d'}$, would be to consider $\mathcal{B}_{d'}(1) = \frac{\alpha_d}{2}(\mathcal{B}_{d'}(0) + \mathcal{G}_{d'}(0))$ and $\mathcal{G}_{d'}(1) = \frac{\alpha_d}{2}\mathcal{B}_{d'}(0) + (\beta_d - \frac{\alpha_d}{2})\mathcal{G}_{d'}(0)$.

Now we prove that

$$\frac{\mathcal{C}_{d'}(1)}{\mathcal{C}_{d'}(0)} \leq \frac{\alpha_{d'}^2 + \beta_{d'}^2}{\alpha_{d'} + \beta_{d'}} \tag{4.3}$$

for all the cases in which the total cost of $\mathcal{J}_d$ follows the worst case scenario.
We have that

$$\begin{aligned}
\frac{\mathcal{C}_{d'}(1)}{\mathcal{C}_{d'}(0)} &= \frac{\alpha_{d'}\mathcal{B}_{d'}(1) + \beta_{d'}\mathcal{G}_{d'}(1)}{\alpha_d\mathcal{B}_{d'}(0) + \beta_d\mathcal{G}_{d'}(0)} \\
&= \frac{\alpha_{d'}\left(\alpha_d \cdot \mathcal{B}_{d'}(0)/2 + \alpha_d \cdot \mathcal{G}_{d'}(0)/2\right)}{\alpha_d\mathcal{B}_{d'}(0) + \beta_d\mathcal{G}_{d'}(0)} \\
&\quad + \frac{\beta_{d'}\left(\alpha_d \cdot \mathcal{B}_{d'}(0)/2 + \beta_d\mathcal{G}_{d'}(0) - \alpha_d \cdot \mathcal{G}_{d'}(0)/2\right)}{\alpha_d\mathcal{B}_{d'}(0) + \beta_d\mathcal{G}_{d'}(0)} \\
&= \frac{\alpha_{d'}\left(\mathcal{C}_{d'}(0)/2 - \beta_d \cdot \mathcal{G}_{d'}(0)/2 + \alpha_d \cdot \mathcal{G}_{d'}(0)/2\right)}{\alpha_d\mathcal{B}_{d'}(0) + \beta_d\mathcal{G}_{d'}(0)} \\
&\quad + \frac{\beta_{d'}\left(\mathcal{C}_{d'}(0)/2 - \alpha_d \cdot \mathcal{G}_{d'}(0)/2 + \beta_d \cdot \mathcal{G}_{d'}(0)/2\right)}{\alpha_d\mathcal{B}_{d'}(0) + \beta_d\mathcal{G}_{d'}(0)} \\
&= \frac{\alpha_{d'} + \beta_{d'}}{2} + \frac{(\alpha_{d'} - \beta_{d'}) \cdot (\alpha_d - \beta_d)\, \mathcal{G}_{d'}(0))}{2(\alpha_d\mathcal{B}_{d'}(0) + \beta_d\mathcal{G}_{d'}(0))}
\end{aligned}$$

Since

$$\frac{\alpha_{d'}^2 + \beta_{d'}^2}{\alpha_{d'} + \beta_{d'}} - \frac{\alpha_{d'} + \beta_{d'}}{2} = \frac{(\alpha_{d'} - \beta_{d'})^2}{2 \cdot (\alpha_{d'} + \beta_{d'})}$$

and $\alpha_{d'} > \beta_{d'}$, Inequality 4.3 is satisfied if

$$\frac{(\alpha_d - \beta_d) \cdot \mathcal{G}_{d'}(0)}{\alpha_d\mathcal{B}_{d'}(0) + \beta_d\mathcal{G}_{d'}(0)} \leq \frac{\alpha_{d'} - \beta_{d'}}{\alpha_{d'} + \beta_{d'}}.$$

The relation $(d-1) \cdot \alpha_d = d \cdot \beta_d$, holds for every $d \geq 2$, so the inequality becomes

$$\frac{\mathcal{G}_{d'}(0)}{d \cdot \mathcal{B}_{d'}(0) + (d-1) \cdot \mathcal{G}_{d'}(0)} \leq \frac{1}{2d' - 1} \Rightarrow (2d' - d) \cdot \mathcal{G}_{d'}(0) \leq d \cdot \mathcal{B}_{d'}(0).$$

Since $2d' - d < d$ and $\mathcal{G}_{d'}(0) \leq \mathcal{B}_{d'}(0)$ the inequality is proven.

The sequence follows at the worst case the recursion established in Lemma 2.13, so we may use the inequalities established in the proof of Lemma 2.14 to prove the cases of $C_{d'}(l+1)/C_{d'}(l)$ with $l \geq 1$. For with $l \geq 4$ there is nothing to prove since the inequality $(\alpha_{d'} - 2\beta_{d'}) \cdot C_{d'}(\ell - 3) \leq \alpha_{d'}(\alpha_{d'} - \beta_{d'}) \cdot C_{d'}(\ell - 4)$ always holds as explained in Lemma 2.14. The case $C_{d'}(3)/C_{d'}(2)$ is proved as equivalent to $C_{d'}(1)/C_{d'}(0) \leq \alpha_{d'}$, which holds, since $C_{d'}(1)/C_{d'}(0) \leq (\alpha_{d'}^2 + \beta_{d'}^2)/(\alpha_{d'} + \beta_{d'}) \leq \alpha_{d'}$.

Now for the last two cases, by the definition of the eliminating average cost, we can always consider the ratio $\mathcal{C}_{d'}(1)/\mathcal{C}_{d'}(0) = (\alpha_{d'}^2 + \beta_{d'}^2)/(\alpha_{d'} + \beta_{d'})$ as in

Lemma 3.3. With that modification the inequalities established for $C_{d'}(2)/C_{d'}(1)$ and $C_{d'}(4)/C_{d'}(3)$ in Lemma 2.14 remain the same.

□

Lemma 4.5. The asymptotic effect for the eliminating average cost of paths $\mathcal{J}_{d-h+1}$ is always smaller than the asymptotic effect of $(2d-1, 0)$ vertices in the case of $d-$orientations.

Proof. The asymptotic effect in the case of paths is $\mathcal{D}(d-h+1)^{\frac{d-1}{2(d-h)-1}}$. First we prove that this holds for $h = 1$. Recall that $\alpha_d/\beta_d = d/(d-1)$.

$$\frac{\mathcal{D}(d)^{2d-2}}{\mathcal{C}_d(2d-1,0)^{2d-3}} < 1 \iff$$
$$\frac{(\alpha_d^2+\beta_d^2)^{2d-2}}{(\alpha_d+\beta_d)^{4d-5}} < 1 \iff$$
$$\left(\frac{\alpha_d^2+\beta_d^2}{(\alpha_d+\beta_d)^2}\right)^{2d-2} \cdot (\alpha_d+\beta_d) < 1 \iff$$
$$\left(1-\frac{2\alpha_d\beta_d}{(\alpha_d+\beta_d)^2}\right)^{2d-2} \cdot (\alpha_d+\beta_d) < 1 \iff$$
$$\left(1-\frac{2d\cdot(d-1)}{(2d-1)^2}\right)^{2d-2} \cdot \binom{2d-1}{d} < 1 \iff$$
$$\left(\frac{2d^2-2d+1}{(2d-1)^2}\right)^{2d-2} \cdot \binom{2d-1}{d} < 1$$

which holds for $d = 2$. So we need to show that the following function is monotonically decreasing for $d \geq 2$.

$$A(d) = \left(\frac{2d^2-2d+1}{(2d-1)^2}\right)^{2d-2} \cdot \binom{2d-1}{d}$$

We have that

$$\frac{A(d+1)}{A(d)} = \frac{\left(\frac{2(d+1)^2-2(d+1)+1}{(2d+1)^2}\right)^{2d} \cdot \binom{2d+1}{d+1}}{\left(\frac{2d^2-2d+1}{(2d-1)^2}\right)^{2d-2} \cdot \binom{2d-1}{d}}$$
$$= \frac{\left(\frac{2(d+1)^2-2(d+1)+1}{(2d+1)^2}\right)^{2d} \cdot 2(2d+1)}{\left(\frac{2d^2-2d+1}{(2d-1)^2}\right)^{2d-2} \cdot (d+1)}$$
$$= \left(\frac{8d^4-2d^2-2d+1}{8d^4-2d^2+2d+1}\right)^{2d-2} \cdot \frac{2}{(2d+1)^3\cdot(d+1)}$$

which is obviously less than 1 for all $d > 0$.

Now, since this inequality holds in dimension d, we use the fact that in smaller dimensions

$$\mathcal{D}(d-h+1)^{\frac{d-h-1}{2(d-h)+1}} \leq \mathcal{C}_{d-h+1}(2(d-h+1)-1,0)^{1/2}$$

for $h \geq 2$. This means that $\mathcal{D}(d-h+1)^{\frac{d-h-1}{2(d-h)+1}}$ is bounded using similar inequalities as in Equation 4.2 from Lemma 4.2. $\square$

Finally, we prove that the asymptotic effect of non-composite $(2(d-h), h)$ vertices is maximized for $h = d-1$.

Lemma 4.6. The following ratio is bigger than 1 for $1 \leq h \leq d-2$.

$$\frac{2^{2(d-h)-1}}{\mathcal{C}_d(2(d-h), h)}$$

Proof. Let us denote $d^* = d-h$ as in Lemma 4.2 and by $S(d^*)$ the above ratio. Since the shift $d^* \to d^*+1$ corresponds to the shift $h \to h-1$. Taking the ratio

$$\frac{S(d^*+1)}{S(d^*)} = 1 + \frac{1}{2d^*+1},$$

one deduces that $S(d^*)$ is clearly increasing. Since $S(2) = 8/6$, the condition holds. $\square$

Now we are ready to prove the bound on d-orientations.

Theorem 4.7. The number of $d-$orientations for a connected d-pseudograph with n vertices and k hanging edges is bounded by

$$\binom{2d-1}{d}^{(n+\frac{1}{d-1})/2} \cdot \left(\frac{2}{\binom{2d-1}{d}^{\frac{1}{d-1}}} \right)^k \tag{4.4}$$

Proof. Let us list the basic categories of vertices to be eliminated and provide a notation for their cardinalities.

- t_d vertices with degree profile $(2d-1, 0)$ or $(2d-1, 1)$ are eliminated.
- s_r vertices with degree profile (r, h), such that their asymptotic effect is strictly smaller than the target bound. For these vertices we consider an upper bound for the cost of their elimination omitting h from the cost function:

  $$\mathcal{C}_d(r) = \max_{\substack{0 \leq h \leq d \\ h \neq (r-2d)/2}} \mathcal{C}_d(r, h).$$

  This definition allows us to use the eliminating average cost for path bounds. Remark that the condition $h \neq (r-2d)/2$ applies only in the case of vertices with even normal degree.
- ℓ_h vertices with normal degree $2(d-h)$, for $1 \leq h \leq d-1$ eliminated with path elimination step.
- m_h vertices with degree profile $(2(d-h), h)$, such that their asymptotic effect is bigger than the target bound.

The total cost of the elimination process is bounded by

$$\mathcal{C}_d(2d-1,0)^{t_d} \cdot \prod \mathcal{C}_d(r)^{s_r} \cdot \prod \mathcal{D}(d-h+1)^{\ell_h} \cdot \prod \mathcal{C}_d(2(d-h),h)^{m_h} \quad (4.5)$$

By Lemma 2.6 the elimination process stops when tree condition $(d-1)\cdot n'+1 = k'$ is achieved, where n' and k' denote the number of vertices and hanging edges at this instance. This means that

$$\begin{aligned} n' &= n - t_d - \sum s_r - \sum \ell_h - \sum m_h \\ k' &= k + (d-1)\cdot t_d + \sum(r-d)\cdot s_r - \sum(d-2h)\cdot \ell_h - \sum(r-d)\cdot m_h \end{aligned}$$

were k' is derived by applying the H.E.E. formula (see Lemma 2.7). These equations combined with tree condition lead to the following inequality on t_d:

$$t_d \leq \frac{n}{2} - \frac{1 + k - \sum(r-1)\cdot s_r - \sum(2(d-h)-1)\cdot \ell_h - \sum(r-1)\cdot m_h}{2d-2}$$

Applying this inequality to Equation 4.5, it is deduced that the following quantity bounds the number of orientations

$$\mathcal{C}_d(2d-1,0)^{\frac{n}{2}-\omega_0} \cdot \prod \left(\frac{\mathcal{C}_d(r)}{\mathcal{C}(2d-1,0)^{\omega_1}}\right)^{s_r} \cdot \prod \left(\frac{\mathcal{D}(d-h+1)}{\mathcal{C}(2d-1,0)^{\omega_2}}\right)^{\ell_h} \cdot \prod \left(\frac{\mathcal{C}_d(2(d-h),h)}{\mathcal{C}(2d-1,0)^{\omega_2}}\right)^{m_h},$$

where $\omega_0 = \frac{k-1}{2d-2}$, $\omega_1 = \frac{r-1}{2d-2}$, and $\omega_2 = \frac{2(d-h)-1}{2d-2}$. Since the terms the asymptotic effect of vertices $(2d-1,0)$ is bigger than the asymptotic effect of vertices in the first product and paths in the second product (see Lemmata 4.2 and 4.3) and the asymptotic effect of non composite $(2, d-1)$ vertices is the biggest among the cases of vertices with asymptotic effect exceeding the target bound (see Lemma 4.6), we deduce that the orientations are bounded by

$$\mathcal{C}(2d-1,0)^{\frac{n}{2}-\frac{k-1}{2d-2}} \left(\frac{2}{\mathcal{C}(2d-1,0)^{\frac{1}{2d-2}}}\right)^{\sum m_h}.$$

By the definition of non-composite vertices with asymptotic effect bigger than the target bound, we have that $\sum m_h$ is bounded by the initial number of hanging edges k. Thus, the bound in Equation 4.4 follows. ☐

d	2	3	4	5	6	7	8
Bézout	4	8	16	32	64	128	256
BES	4.89	8.94	16.7	31.7	60.7	117.1	226.8
BEV	3.78	6.84	12.68	23.89	45.53	87.46	168.9
new	3.46	6.32	11.83	22.44	42.98	82.84	160.4

Table 4.1: *The base to the power of $|V|$ for the asymptotic bound in dimensions $2 \leq d \leq 8$.* Bézout *denotes the Bézout bound,* BES *is the bound derived in [3] using matrix permanents,* BEV *is the bound derived in [5] that uses elimination techniques to bound outdegree constrained orientations, and* new *is the bound derived in this paper. Note that as we go on higher dimensions, the difference between our bound and the next best increases.*

Lemma 4.8. Let G be a minimally rigid graph in dimension d. There is a fixed subgraph $K_{d'}$ in G, with $d' < d$, that its removal does not break the connectivity of G.

Proof. The graph G is at least d-connected. Let S be the minimum separator of G. Note that $|S| \geq d$. Hence the removal of any subgraph of G with less than d vertices cannot break its connectivity. □

Now we are ready to prove Theorem 4.1.

Proof. First we prove that $\dfrac{2^{d-1}}{\mathcal{C}(2d-1,0)} < 1$. Observe that the ratio

$$\frac{2^d \cdot \binom{2d-1}{d}}{2^{d-1} \cdot \binom{2d+1}{d+1}} = \frac{d+1}{2d+1} < 1$$

for all $d \geq 2$. This implies that $2^{d-1} < \mathcal{C}(2d-1,0)$ holds for every $d \geq 2$, since for $d = 2$ we have that $2 < \binom{3}{2}$.

Lemma 4.8 indicates that there is always at least a fixed K_2 that is not a separator for $d \geq 3$. The bound results applying Theorem 4.1 for $d-$orientations to Theorem 2.3. □

CHAPTER 5

ON THE MAXIMAL NUMBER OF *mBe*.

In this section we provide examples of Laman graphs with maximal number of 2-orientations, among certain cases we computed. Let us remark that an exhaustive search over all Laman graphs with big number of vertices is almost infeasible, since their number is gigantic. For that reason, graphs were constructed by using repetitive Henneberg steps [17] to Laman graphs with big embedding number (data for the embedding numbers was found in [11]). To provide these bounds, we calculate them using the code from [6].

Given a Laman graph $G(V, E)$, we denote with $\mu(G)$ the number of its complex embeddings. Recall that $mBe(G, e)$ denotes the bound derived from Theorem 2.3 for a fixed edge $e \in E$ and $\mathcal{R}(G, e)$ the corresponding number of orientations.

Then applying Theorem 3.6, the following inequality holds:

$$\mu(G) \leq \min_{e \in E}(mBe(G, e)) \ \leq mBe(G, e) \ \leq \frac{16}{3^{7/2}} \cdot \left(2 \cdot 3^{1/2}\right)^{|V|-2}$$

since the bound can vary for different fixed edges.

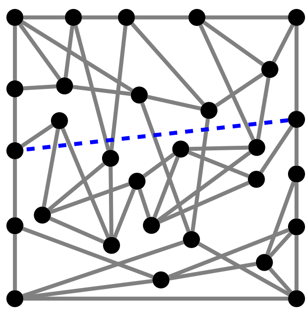

Figure 5.1: *The graph G_{29a} is a Laman graph on 29 vertices. The dashed blue edge corresponds to the fixed edge that yields the maximal m-Bézout. Note that $k = 6$ for the specific fixed edge.*

In order to compute the asymptotic order of our bound, we consider only the non-fixed vertices as in [2, 11]. This choice is made due to the underlying algebraic systems that lead to the bound in [3]. We also exploit the fixed terms to examine if the base of the exponent is increased. This is done both in the case of the bound on embeddings from Theorem 3.6 and the bound on orientation from Theorem 3.4. More precisely, to compute the asymptotic order of $mBe(G, e)$, we consider $mBe(G, e)^{1/(|V|-2)}$.

In order to include the fixed term of the bound, we need to take into account that a part of the fixed term is related to the target bound for the orientations which is $3^{1/2} \approx 1.7321$. So we consider the following equality

$$mBe(G, e) = \frac{16}{a^7} \cdot (2 \cdot a)^{|V|-2}. \tag{5.1}$$

Finally, the deletion of certain fixed edges may result to connected d-pseudographs with more than 4 hanging edges, which is the default value used for Theorem 3.6. Thus, in order to examine only the bound on orientations, we use also the relation

$$\mathcal{R}(G, e) = \frac{2^k}{\alpha_k^{2k-1}} \cdot a_k^{|V|-2}. \tag{5.2}$$

The graph G_{29a} is a $29-$vertex graph and has the maximum asymptotic bound we have computed (see Figure 5.1). For the edge e that maximizes this bound, we have that $mBe(G_{29a}, e) = 21, 947, 282, 882, 560$, so $\mathcal{R} = 163, 520$. The asymptotic order for these numbers, without taking into consideration the constant term are respectively $mBe(G_{29a}, e)^{1/(|V|-2)} = 3.1198$ and $\mathcal{R}(G_{29a}, e)^{1/(|V|-2)} = 1.5599$. Including the constant term and considering the default value $k = 4$, we get $a = 1.5866$. Finally, taking into account that both the fixed vertices have degree 4, we compute that taking into account $k = 6$, we get $a_6 = 1.6329$. We believe that with bigger computational resources, these numbers can be further increased.

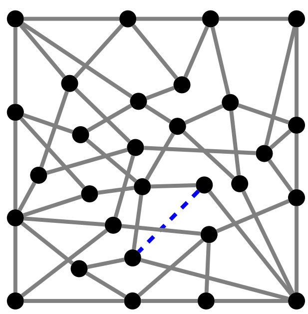

Figure 5.2: *The graph G_{29b} is also a Laman graph on 29 vertices and the dashed blue edge corresponds to the fixed edge that yields the minimum m-Bézout. For that fixed edge $k = 5$.*

Although this graph has a big bound for this specific edge, its minimum bound is much lower, namely $416, 611, 827, 712$ with an asymptotic order of $2.6938^{|V|}$. Thus, we ran computations aiming for graphs with big minimum bound.

The graph G_{29b} is the 29-vertex graph with the maximum minimal asymptotic bound we have computed (see Figure 5.2). For the edges e that minimizes the bound for the specific graph, we have that $mBe(G_{29b}, e) = 784, 502, 620, 160$, which yields

$mBe(G_{29b}, e)^{1/(|V|-2)} = 2.7576$ as an asymptotic order. Divided by 2, we have that the asymptotic order for the orientations is $\mathcal{R}(G_{29b}, e)^{1/(|V|-2)} = 1.3788$. Setting the default value of hanging edges, we get $a = 1.3431$, while using the fact that actually $k = 5$, the result is $a_5 = 1.3355$. Notice that a and a_5 in that case are smaller than the asymptotic order of $\mathcal{R}(G_{29b}, e)$. That happens because the square of all these values is strictly smaller than 2 (so the fraction in Equations 5.1 and 5.2 is smaller than 1).

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

In this thesis we have managed to improve the upper bounds on the embedding number of minimally rigid graphs, via bounding the number of outdegree constrained orientation for certain cases. This bound is still far from the lower bounds on the maximal number of complex embeddings in the cases of dimension 2 and 3, but as we show in Section 5, it seems to be close to the bound on the orientations for Laman graphs. Thus, we consider that the upper bound on orientations may be sharp, but demanding computations are needed to verify this conjecture.

More demanding computations are also required for improving the lower bounds on the embedding number that may reduce the gap between lower and upper bounds. In the case that the bound is indeed sharp on orientations, but loose for the embedding number, one may consider other ways to improve upper bounds. One idea is to examine if there are certain combinatorial properties for the fixed K_d whose removal minimizes the number of orientations. Another approach would be to consider tools from sparse algebraic geometry on determinantal varieties of Cayley-Menger matrices or even to use a completely different formulation to represent the problem. A more efficient stopping condition might improve the asymptotic cost of the elimination process. By taking under consideration, the graph density and the average degree of a graph, one may improve the upper bounds.

Finally, our bounds may be extended generally for all outdegree constrained d-orientations. In that case the restriction on degree is not applied for $(2d-1)-$valent vertices, but for $(2d)$-valent vertices. This result may also have applications on the multihomogeneous bound of certain polynomial systems, as demonstrated in [4].

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