Lexicographic Sets

Evangelos Protopapas
AL1180013

Examination committee:
Dimitrios Thilikos, Department of Mathematics, National and Kapodistrian University of Athens.
Stavros Kolliopoulos, Department of Informatics and Telecommunications, National and Kapodistrian University of Athens.
Panos Rondogiannis, Department of Informatics and Telecommunications, National and Kapodistrian University of Athens.

Supervisor:
Panos Rondogiannis, Professor, Department of Informatics and Telecommunications, National and Kapodistrian University of Athens.

Μεταπτυχιακό Πρόγραμμα «Αλγόριθμοι, Λογική και Διακριτά Μαθηματικά» – 2016

Η παρούσα Διπλωματική Εργασία
εκπονήθηκε στα πλαίσια των σπουδών
για την απόκτηση του
Μεταπτυχιακού Διπλώματος Ειδίκευσης
«Αλγόριθμοι, Λογική και Διακριτά Μαθηματικά»
που απονέμει το
Τμήμα Πληροφορικής και Τηλεπικοινωνιών
του
Εθνικού και Καποδιστριακού Πανεπιστημίου Αθηνών

Εγκρίθηκε την .... από Εξεταστική Επιτροπή
αποτελούμενη από τους:

Ονοματεπώνυμο	Βαθμίδα	Υπογραφή
1.	. . . .	. . . .
2.	. . . .	. . . .
3.	. . . .	. . . .

ABSTRACT

The purpose of this paper is to investigate whether a recently proposed infinite-valued logic can lead to a novel, non-classical set theory. As in fuzzy set theory, the members of a set in the proposed theory may have different degrees of participation, expressed by different levels of truth values. But unlike in fuzzy set theory, the subset relation as well as the union and intersection operations are defined in a lexicographic way with respect to the truth values of elements in the involved sets. That is why we call the sets of the proposed theory, lexicographic sets. We prove that many known properties that apply to the classical set theory still apply to lexicographic sets. In addition, we give indications that the proposed theory may have practical applications in areas of Computer Science where the lexicographic relation plays a key role. More specifically, using the proposed theory, we prove a generalized model intersection theorem for logic programs with negation.

ΣΥΝΟΨΗ

Ο στόχος της εργασίας αυτής είναι να διερευνήσει αν μια πρόσφατα προταθείσα απειρότιμη λογική μπορεί να οδηγήσει σε μια νέα, μη κλασική θεωρία συνόλων. Όπως και στην ασαφή θεωρία συνόλων (fuzzy set theory), τα μέλη ενός συνόλου στην προτεινόμενη θεωρία μπορούν να έχουν διαφορετικό βαθμό συμμετοχής που εκφράζεται με διαφορετικά επίπεδα τιμών αλήθειας. Σε αντίθεση όμως με την ασαφή θεωρία συνόλων, η σχέση υποσυνόλου καθώς και οι πράξεις της ένωσης και της τομής, ορίζονται με λεξικογραφικό τρόπο σε σχέση με τις τιμές αλήθειας των στοιχείων των εμπλεκόμενων συνόλων. Για το λόγο αυτό ονομάζουμε τα σύνολα της προτεινόμενης θεωρίας, λεξικογραφικά σύνολα (lexicographic sets). Αποδεικνύουμε ότι πολλές γνωστές ιδιότητες που ισχύουν στην κλασική θεωρία συνόλων εξακολουθούν να ισχύουν και για τα λεξικογραφικά σύνολα. Επιπλέον, δίνουμε ενδείξεις ότι η προτεινόμενη θεωρία μπορεί να έχει πρακτικές εφαρμογές σε περιοχές της Πληροφορικής όπου η λεξικογραφική διάταξη παίζει καθοριστικό ρόλο. Πιο συγκεκριμένα, χρησιμοποιώντας την προτεινόμενη θεωρία, αποδεικνύουμε ένα γενικευμένο θεώρημα τομής μοντέλων για λογικά προγράμματα με άρνηση.

CONTENTS

CHAPTER 1 INTRODUCTION

1.1 Motivation

The motivation for this thesis begins with the study of the declarative semantics of logic programs. In the following we assume that the reader is familiar with partially ordered sets and particularly lattices (e.g. [1]) as well as with some basic notions regarding logic programming, such as "atoms", "literals", "clauses", "head/body of a rule", "ground instantiations", "Herbrand Base", "Herbrand interpretations/models", "definite programs", "negation-as-failure", etc. For an introduction to these notions the reader is redirected to [5].

The declarative semantics of classical logic programs (i.e. definite programs, where negation is not present in the body of any rule) was developed by van Emden and Kowalski in [7], and it is the result of an elegant fixed point theory that is mainly attributed to the least fixed point theorem of Kleene as well as a weaker version of the well known Knaster-Tarski theorem. The main concept of these semantics is as follows. Let P be a definite program. Let B_P denote the Herbrand Base of this program. The semantics is based on the fact that the set 2^{B_P}, which is the set of all Herbrand interpretations of the program, is a complete lattice under the partial order of set inclusion $\subseteq$. The least upper bound of any set of Herbrand interpretations is the union of all the interpretations in the set and symmetrically the greatest lower bound is the intersection. The entire semantics can be summed up in two major points. Firstly, the famous Model Intersection Theorem states that the intersection of all Herbrand models of P is a model of P called the *least Herbrand model* of P. This model is usually denoted by M_P. Secondly, M_P is also characterized using fixed point techniques. An operator $T_P : 2^{B_P} \mapsto 2^{B_P}$ is defined, which is called the immediate consequence operator, which is monotonic as well as continuous. It is then proved using the aforementioned fixed point theorems that T_P has a *least fixed point* which coincides with the least model M_P. It is also proved that M_P is precisely the set of ground atoms which are logical consequences of P and thus it is justifiably considered the intended model for P.

Unfortunately after introducing negation to logic programming, the classical tools can no longer be used to define the semantics of logic programs. The major issue lies in the fact that the immediate consequence operator T_P loses its monotonicity in this expanded formalism. Various attempts have been made to give semantics to logic programs with negation. However most of these are not purely model theoretic in the sense that the meaning of a program can not be computed just by considering its set of models. That was the case until Rondogiannis and Wadge in [6], gave the first purely model-theoretic characterization of the semantics of logic programs with negation-as-failure allowed in clause bodies. In these semantics the meaning of a program is, just as in the classical case, the unique *minimum model* in a novel ordering of a program's interpretations.

The basic idea behind their approach, namely the *infinite-valued semantics*, is that in order to obtain a minimum model semantics for logic programs with negation, it is necessary to consider a refined multiple-valued logic which will allow the meaning of negation-as-failure to be expressed properly. The logic of [6] contains one F_α and one T_α for each countable ordinal α, and also an intermediate truth value denoted by 0. The ordering of the truth values is as follows:

$$F_0 < F_1 < \cdots < F_\omega < \cdots F_\alpha < \cdots < 0 < \cdots < T_\alpha < \cdots < T_\omega < \cdots < T_1 < T_0$$

F_0 and T_0 represent the classical *False* and *True* values, while 0 is the *undefined* value. With this refined set of values, atoms in B_P are no longer treated as being either True or False. The ordinal indices of truth values now correspond to the level at which truth or falsity holds. The definition of interpretations changes as well. An interpretation of a program is extended to a function mapping atoms to truth values. Then, intuitively negation-as-failure is interpreted as follows. The more times an atom is negated the closer it approaches the intermediate value 0. Finally a more refined ordering on interpretations is used extending the subset $\subseteq$ relation that is used in the classical treatment. This new ordering denoted by $\sqsubseteq_\infty$, is a lexicographic-like relation that intuitively gives a higher priority to atoms belonging in the lower stages (smaller ordinal indices) than those belonging to upper stages. It is proven indirectly in [9], that the set of all infinite-valued interpretations ordered by $\sqsubseteq_\infty$ actually forms a complete lattice as in the classical case. Finally using this extended formalism a Model Intersection Theorem is proved. This theorem defines a procedure which starts with the set of all infinite-valued models of a program and outputs a single model which is the *minimum model* with respect to $\sqsubseteq_\infty$. Also an immediate consequence operator T_P is defined, which naturally extends the classical definition. It is then proved that this T_P does have a *least fixed point* which coincides with the model produced by the model intersection theorem, even though the operator itself is not monotonic with respect to $\sqsubseteq_\infty$.

The infinite-valued semantics gives rise to the ideas that we will be examining in this thesis. The basic idea stems from the fact that the relation $\sqsubseteq_\infty$ seems to generalize the classical subset relation $\subseteq$ in the infinite-valued setting. Both the set of infinite-valued models paired with $\sqsubseteq_\infty$ and the set of classical Herbrand models paired with $\subseteq$ form complete lattices. Thus the following questions arise. Can we abstract away from the logic programming specific definitions and define an extension of the classical notion of set that looks like an infinite-valued interpretation? Also can we define a lexicographic ordering like $\sqsubseteq_\infty$ to be used as the subset relation for our novel extended concept of a set? How would classical operations like the union, intersection and complement look like in this extended setting? What properties might they possess? Will the classical set theoretic properties hold in the extended setting?

1.2 Brief history of set extensions

Various extensions to the classical definition of a set have been introduced throughout the 20th century. The main goal of these extensions is to define mathematical frameworks that can treat imprecision and ambiguity present in data. The oldest and most well-known such extension is the *fuzzy set* concept that was introduced in parallel and independently in 1965 by Lotfi A. Zadeh in [8] and Dieter Klaua. Dieter Klaua introduced the concept of a *many-valued set*. These many-valued sets had the fuzzy sets of Zadeh as a particular case. Zadeh's approach has been more influential, so from now on we will focus on Zadeh's *fuzzy set theory*.

Formally, in a given universe of elements X, a fuzzy set A in X is characterized by a *membership function* $f_A(x)$, which maps each element $x \in X$ to a real number in the interval $[0, 1]$. The value of $f_A(x)$ represents the "grade of membership" of x in A. The nearer the value of $f_A(x)$ to unity, the higher the grade of membership of x in A.

A generalization of the notion of fuzzy set was introduced in 1967 by Joseph Goguen, who was a student of Zadeh, in [3]. This generalization is usually called an *L-fuzzy set*. L-fuzzy sets generalize fuzzy sets by considering order structures beyond the unit interval. An L-fuzzy set generalizes Zadeh's membership function to an arbitrary order structure L and formally an L-fuzzy set is a function of the form $A\colon X \mapsto L$. Usually this order structure is required to be a partially ordered set. Also since asking what the maximum and minimum values of a fuzzy set are seems rather important, it is more commonly required that the partially ordered set is at least a complete lattice.

There are many more concepts similar to or more general than fuzzy sets. Some are special cases of fuzzy sets that add more constraints or change the membership function. Others are entirely novel theories that try to mathematically model imprecision and uncertainty in different ways. For a detailed introduction to such theories the reader is redirected to [4].

The reader might have already suspected that the *lexicographic set* concept that we develop in this thesis is a special case of the general *L-fuzzy set*. That is indeed the case. In our case the order structure L will be the truth values used in the infinite-valued semantics. But there is one important difference. In general, for *L-fuzzy sets*, it is argued that, by considering a *point-wise* relation to compare elements in L^X, many special laws which holds for L will extend to L^X. Then the focus is shifted to studying the desired structure for L.

In this thesis we take a different approach. While we do consider the point-wise relation as a possible subset relation for our lexicographic sets, our main focus is studying another relation, which we call the *lexicographic subset relation*, which is directly defined on lexicographic sets, and does not try to directly exploit the special structure that the truth values of the infinite-valued semantics might have.

1.3 Thesis Goals

This thesis mainly focuses on answering the questions posed in section 1.1. More specifically,

- We introduce an extension of the classical notion of a set, which we call a *lexicographic set* by expanding the classical bivalent condition that assesses the membership of elements. The lexicographic set is represented by a function mapping the elements of a given universe to an infinite number of possible membership values that will correspond to the truth values used in the infinite-valued semantics. We also introduce appropriate definitions of the lexicographic subset relation, lexicographic union and intersection and lexicographic complement. We also study various extensions of classical set-theoretic properties in this setting.

- We apply the new definitions, providing a proof for the Model Intersection theorem of the infinite-valued approach using just the lexicographic intersection operation. Actually we prove a slightly stronger result which states that the lexicographic intersection of any non-empty collection of infinite-valued models of a program is also a model of the program. Thus, we fully generalize the classical Model Intersection Theorem.

CHAPTER 2

LEXICOGRAPHIC SETS, DEFINITIONS AND PROPERTIES

In this chapter we formally define lexicographic sets, appropriate subset relations, union, intersection and complement definitions as well as examine a few properties they possess.

2.1 Lexicographic sets and basic definitions

Let κ be a limit ordinal. We will not explicitly define it for now. Also consider a non-empty universe X of elements.

Definition 2.1. *Set of truth values.* We define:

$$V = \{F_\alpha \mid \alpha < \kappa\} \cup \{T_\alpha \mid \alpha < \kappa\} \cup \{0\}$$

to be the set of truth values. These truth values are ordered as follows. Consider ordinals $\alpha < \beta < \kappa$ then $F_\alpha < F_\beta$ and $T_\beta < T_\alpha$. Also $(\forall\alpha)\ F_\alpha < 0$ and $(\forall\alpha)\ 0 < T_\alpha$.

We define lexicographic sets as follows:

Definition 2.2. *Lexicographic set.* A lexicographic set is a total function mapping elements of X to truth values in V. The set $\mathcal{X} = V^X$, which is the set of all total functions $A : X \mapsto V$ is the set of all lexicographic sets.

Intuitively each truth value represents the degree at which an element belongs to (T_α) or does not belong to (F_α) a lexicographic set. We use the intermediate truth value 0 to express the concept of "*undefined*" or "*unknown*". A 0 value indicates that the membership status of an element is unknown. E.g. let $X = \{a, b, c, d, e\}$. Then $A = \{\langle a, F_0\rangle, \langle b, F_1\rangle, \langle c, T_2\rangle, \langle d, T_0\rangle, \langle e, 0\rangle\} \in \mathcal{X}$. So a is not a member of A, b is not a member of A but not at the degree that a is not, while c is a member of A but not at the degree that d is a member of A which is a member of A. Also the degree of membership for e is undefined.

We define the order of truth values as follows:

Definition 2.3. *Order of a truth value.* Let $\alpha < \kappa$ be an ordinal. The order of a truth value is defined as: $order(T_\alpha) = \alpha$, $order(F_\alpha) = \alpha$ and $order(0) = \kappa$.

We also define the following:

Definition 2.4. Let $A \in \mathcal{X}$ be a lexicographic set. Let $v \in V$ be a truth value. Then $A \mid\mid v = \{x \in X \mid A(x) = v\}$.

Definition 2.5. *Level of a lexicographic set.* Let $A \in \mathcal{X}$ be a lexicographic set. Let $\alpha \leq \kappa$ be an ordinal. Then $A\#\alpha = \{\langle x, v\rangle \in A \mid order(v) = \alpha\}$.

In the next section we introduce two different definitions for the subset relation of lexicographic sets.

2.2 Subset Relations and Equality

The most obvious way to define the subset relation would be to directly compare the truth values assigned between two lexicographic sets per element in X.

We define the subset relation $\leq$, which we call the *point-wise subset relation* as follows:

Definition 2.6. *Point-wise subset relation*. Let $A, B \in \mathcal{X}$ be two lexicographic sets. Then:

$$A \leq B \leftrightarrow (\forall x \in X)\ A(x) \leq B(x)$$

A not so obvious choice but a choice of great importance for this thesis, would be to use the $\sqsubseteq_\infty$ relation used to order infinite-valued interpretations in [6].

Let $\alpha < \kappa$ be an ordinal. First we define the equality relation $=_\alpha$ of order α.

Definition 2.7. *Equality relation $=_\alpha$ of order $\alpha < \kappa$.* Let $A, B \in \mathcal{X}$ be two lexicographic sets. Then:

$$A =_\alpha B \leftrightarrow (\forall \beta \leq \alpha)\ A\#\beta = B\#\beta$$

Then we define the proper subset relation $\sqsubset_\alpha$ of order $\alpha < \kappa$ as follows:

Definition 2.8. *Proper subset relation $\sqsubset_\alpha$ of order $\alpha < \kappa$.* Let $A, B \in \mathcal{X}$ be two lexicographic sets. Then:

$$A \sqsubset_\alpha B \leftrightarrow \big((\forall \beta < \alpha)\ A =_\beta B\big) \ \wedge\ \big((A \mid\mid T_\alpha \subset B \mid\mid T_\alpha \ \wedge\ A \mid\mid F_\alpha \supseteq B \mid\mid F_\alpha) \vee (A \mid\mid T_\alpha \subseteq B \mid\mid T_\alpha \ \wedge\ A \mid\mid F_\alpha \supset B \mid\mid F_\alpha)\big)$$

Then we define the subset relation $\sqsubseteq_\alpha$ of order $\alpha < \kappa$ as follows:

Definition 2.9. *Subset relation $\sqsubseteq_\alpha$ of order $\alpha < \kappa$.* Let $A, B \in \mathcal{X}$ be two lexicographic sets. Then:

$$A \sqsubseteq_\alpha B \leftrightarrow A \sqsubset_\alpha B \ \vee\ A =_\alpha B$$

Finally we define the subset relation $\sqsubseteq$, which we call the *lexicographic subset relation* as follows:

Definition 2.10. *Lexicographic subset relation*. Let $A, B \in \mathcal{X}$ be two lexicographic sets. Then:

$$A \sqsubseteq B \leftrightarrow \big((\exists \alpha < \kappa)\ A \sqsubset_\alpha B\big) \ \vee\ A = B$$

2.2.1 Equality

In this subsection we define lexicographic set equality as well as provide a different but equivalent characterization based on the equality relation $=_\alpha$ of order $\alpha < \kappa$, for when two lexicographic sets are equal.

Definition 2.11. *Lexicographic set equality*. Let $A, B \in \mathcal{X}$ be lexicographic sets. We say that A and B are *equal*, written as $A = B$, if and only if $A(x) = B(x)$.

Lemma 2.12. Let $A, B \in \mathcal{X}$ be lexicographic sets. Then:

$$A = B \leftrightarrow (\forall \alpha < \kappa)\ A =_\alpha B$$

Proof. If $A = B$ holds then trivially $(\forall \alpha < \kappa)\ A =_\alpha B$ also holds. If $(\forall \alpha < \kappa)\ A =_\alpha B$ holds, then by definition of $=_\alpha$, for any $x \in X$ such that $order(A(x)) < \kappa$, we have that $A(x) = B(x)$. Then observe that for any $x \in X$ such that $order(A(x)) = \kappa$, since both A, B are total functions it must hold that $A(x) = 0 = B(x)$. Then $A = B$. □

Throughout this thesis we use the two equality definitions interchangeably without further explanation.

2.2.2 A few properties

In [6] it is proven for infinite-valued models that $I \leq J \to I \sqsubseteq_\infty J$, where I, J are infinite-valued interpretations. The proof still holds in this context and thus we have the following lemma.

Lemma 2.13. Let $A, B \in \mathcal{X}$ be two lexicographic sets. Then:

$$A \leq B \to A \sqsubseteq B$$

The opposite does not hold. For example consider $X = \{a, b\}$. Then let $A = \{\langle a, F_0\rangle, \langle b, T_1\rangle\}$ and $B = \{\langle a, T_0\rangle, \langle b, F_1\rangle\}$. It is easily seen that $A \sqsubseteq B$ but $A \nleq B$.

We prove the following lemma about the subset relation $\sqsubseteq_\alpha$ of order $\alpha < \kappa$.

Lemma 2.14. The subset relation $\sqsubseteq_\alpha$ of order $\alpha < \kappa$ is a preorder.

Proof. Let $A, B, C \in \mathcal{X}$ be lexicographic sets.

First since $A =_\alpha A$, trivially $A \sqsubseteq_\alpha A$ holds and $\sqsubseteq_\alpha$ is reflexive.

Next assume that $A \sqsubseteq_\alpha B$ and $B \sqsubseteq_\alpha C$. By definition of $\sqsubseteq_\alpha$ we have that for all ordinals $\beta < \alpha$, $A =_\beta B$ and $B =_\beta C$. By definition of $=_\beta$ we have that $A\#\beta = B\#\beta$ and $B\#\beta = C\#\beta$ which implies that $A\#\beta = C\#\beta$. By definition of $=_\beta$ we have that $A =_\beta C$. It remains to show that $A \mid\mid T_\alpha \subseteq C \mid\mid T_\alpha$ and that $A \mid\mid F_\alpha \supseteq C \mid\mid F_\alpha$. Since $A \sqsubseteq_\alpha B$ we have that $A \mid\mid T_\alpha \subseteq B \mid\mid T_\alpha$ and $A \mid\mid F_\alpha \supseteq B \mid\mid F_\alpha$. Since $B \sqsubseteq_\alpha C$ we have that $B \mid\mid T_\alpha \subseteq C \mid\mid T_\alpha$ and $B \mid\mid F_\alpha \supseteq C \mid\mid T_\alpha$. Thus $A \mid\mid T_\alpha \subseteq C \mid\mid T_\alpha$ and $A \mid\mid F_\alpha \supseteq C \mid\mid F_\alpha$ hold and by definition of $\sqsubseteq_\alpha$ we have that $A \sqsubseteq_\alpha C$. Then $\sqsubseteq_\alpha$ is transitive. □

We also prove the following lemma about the lexicographic subset relation $\sqsubseteq$.

Lemma 2.15. The lexicographic subset relation $\sqsubseteq$ is a partial order.

Proof. Let $A, B, C \in \mathcal{X}$ be lexicographic sets.

First trivially since $A = A$, by definition of $\sqsubseteq$ we have that $A \sqsubseteq A$ and $\sqsubseteq$ is reflexive.

Next assume that $A \sqsubseteq B$ and $B \sqsubseteq A$ while $A \neq B$. Let $\alpha < \kappa$ be the least ordinal such that $A\#\alpha \neq B\#\alpha$. Then $(\forall \beta < \alpha)\ A =_\beta B$. Since $A \sqsubseteq B$, $(\forall \beta < \alpha)\ A =_\beta B$ and $A\#\alpha \neq B\#\alpha$ it must hold that $A \sqsubset_\alpha B$. But then the definition of $\sqsubset_\alpha$ implies that $B \sqsubset_\alpha A$ cannot hold. Also since $\alpha < \kappa$ is the least ordinal such that $A\#\alpha \neq B\#\alpha$ then the the definition of $\sqsubset_\beta$ implies that for any ordinal $\beta < \kappa$, $B \sqsubset_\beta A$ cannot hold. Then by definition of $\sqsubseteq$, $B \sqsubseteq A$ cannot hold which contradicts our assumptions. Thus $A = B$ and $\sqsubseteq$ is antisymmetric.

Finally assume that $A \sqsubseteq B$ and $B \sqsubseteq C$. If $A = B$ then trivially $A \sqsubseteq C$. Else if $B = C$ then trivially $A \sqsubseteq C$. If $A \neq B$ and $B \neq C$ then by definition of $\sqsubseteq$ there exist ordinals $\alpha, \beta < \kappa$ such that $A \sqsubset_\alpha B$ and $B \sqsubset_\beta C$. Let $\gamma = \min\{\alpha, \beta\}$. Then it holds that $A \sqsubseteq_\gamma B$ and $B \sqsubseteq_\gamma C$. By lemma 2.14, $\sqsubseteq_\gamma$ is transitive and thus $A \sqsubseteq_\gamma C$. Finally observe that if $\gamma = \alpha$, $A \sqsubset_\gamma B$, else if $\gamma = \beta$, $B \sqsubset_\gamma C$. In either case, the definition of $\sqsubseteq_\gamma$ implies that $A \neq_\gamma C$. Thus $A \sqsubset_\gamma C$. By definition of $\sqsubseteq$ we have that $A \sqsubseteq C$ and $\sqsubseteq$ is transitive. □

In the next section we introduce two different definitions for the (generalized) union relation of lexicographic sets.

2.3 Union operations

As expected, we define one (generalized) union operation for each of the subset relations $\leq$ and $\sqsubseteq$. A natural way to define the union, that generalizes that of classical set theory, would be to use the least upper bound under these relations.

2.3.1 Point-wise union

We define the (generalized) point-wise union with respect to $\leq$ as follows:

Definition 2.16. *Point-wise union*. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. We define the point-wise union of $\mathcal{S}$ as the least upper bound $\bigvee \mathcal{S}$ of $\mathcal{S}$ under $\leq$. Let $x \in X$, then:

$$\left(\bigvee \mathcal{S}\right)(x) = lub\{A(x) \mid A \in \mathcal{S}\}$$

Remark. If $\mathcal{S} = \emptyset$ then $(\forall x \in X)\ \left(\bigvee \mathcal{S}\right)(x) = F_0$, since by definition of lub we have that $lub\ \emptyset = F_0$. As we will mention in a later section this is the $\leq$-minimum element in $\mathcal{X}$. This coincides with classical set theory where $\bigcup \emptyset = \emptyset$ and $\emptyset$ is the $\subseteq$-minimum element.

Example 2.17. *Let $\kappa = \omega$, let $X = \{a\}$ and $\mathcal{S} = \{A_n \mid n < \omega\}$, where $A_n = \{\langle a, F_n \rangle\}$. Then $\bigvee \mathcal{S} = \{\langle a, lub\{F_n \mid n < \omega\}\rangle\} = \{\langle a, 0\rangle\}$.*

2.3.2 Lexicographic union

In [9] it is proven that the set of infinite-valued interpretations paired with the relations $\sqsubseteq_\alpha$, where $\alpha < \omega_1$, form a model of the Axioms presented in that paper. That proof immediately carries over to our setting. Thus we can use all the theoretical constructs presented in that paper. Of particular interest is the fact that every non-empty set of lexicographic sets $\mathcal{S} \subseteq \mathcal{X}$ has a least upper bound under $\sqsubseteq$ which can be constructed in steps.

Let $\alpha < \kappa$ be an ordinal and $A \in \mathcal{X}$ be a lexicographic set. We define:

$$(A]_\alpha = \{B \in \mathcal{X} \mid (\forall \beta < \alpha)\ A =_\beta B\}$$

Then let $\mathcal{S} \subseteq (A]_\alpha$. We define $\bigsqcup_\alpha \mathcal{S}$ as follows. Let $x \in X$, then:

$$\Big(\bigsqcup_\alpha \mathcal{S}\Big)(x) = \begin{cases} A(x), & \text{If } order(A(x)) < \alpha \\ T_\alpha, & \text{If } (\exists B \in \mathcal{S})\ B(x) = T_\alpha \\ F_\alpha, & \text{If } (\forall B \in \mathcal{S})\ B(x) = F_\alpha \\ F_{\alpha+1}, & \text{otherwise} \end{cases}$$

We now define the following sequence of lexicographic sets.

Definition 2.18. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. We define a sequence of lexicographic sets $(S_\alpha)_{\alpha \leq \kappa}$ as follows. For all $\alpha < k$ we define:

$$\mathcal{S}_\alpha = \{A \in \mathcal{S} \mid (\forall \beta < \alpha)\ A =_\beta S_\beta\}$$

$$S_\alpha = \begin{cases} \bigsqcup_\alpha \mathcal{S}_\alpha, & \text{If } \mathcal{S}_\alpha \neq \emptyset \\ \bigvee_{\beta<\alpha} S_\beta, & \text{If } \mathcal{S}_\alpha = \emptyset \end{cases}$$

Then:

$$S_\kappa = \bigvee_{\alpha<\kappa} S_\alpha$$

Intuitively the above definition defines an infinite number of approximations of the least upper bound of the lexicographic sets in $\mathcal{S}$. As it proceeds through the ordinals $\alpha < \kappa$, each approximation S_α builds upon the previous approximations, improving the approximation, by constructing the α-th level to be an upper bound of the remaining lexicographic sets in $\mathcal{S}_\alpha$, as compact as possible, with respect to the lexicographic subset relation $\sqsubseteq$.

Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\mathcal{S} \sqsubseteq A \leftrightarrow (\forall B \in \mathcal{S})\ B \sqsubseteq A$$

We prove the following lemma.

Lemma 2.19. Let $A \in \mathcal{X}$ be a lexicographic set. Let $\mathcal{S} \subseteq (A]_\alpha$ for ordinal $\alpha < \kappa$. Then:

- $\mathcal{S} \sqsubseteq_\alpha \bigsqcup_\alpha \mathcal{S}$
- $\big(\forall B \in (A]_\alpha\big)\ \mathcal{S} \sqsubseteq_\alpha B \to \big(\bigsqcup_\alpha \mathcal{S} \sqsubseteq_\alpha B \wedge \bigsqcup_\alpha \mathcal{S} \leq B\big)$

Proof. Since $\mathcal{S} \subseteq (A]_\alpha$ it holds that $(\forall B \in \mathcal{S})(\forall C \in \mathcal{S})(\forall \beta < \alpha)\ B =_\beta C$. So by the definition of $\bigsqcup_\alpha$, it holds that $\bigsqcup_\alpha \mathcal{S} =_\beta B =_\beta C$. Then $\bigsqcup_\alpha \in (A]_\alpha$.

Let $B \in \mathcal{S}$. We prove that $B \mid\mid T_\alpha \subseteq \bigsqcup_\alpha \mathcal{S} \mid\mid T_\alpha$ and $B \mid\mid F_\alpha \supseteq \bigsqcup_\alpha \mathcal{S} \mid\mid F_\alpha$. Let $x \in X$ such that $B(x) = T_\alpha$. By definition of $\bigsqcup_\alpha$ it holds that $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = T_\alpha$. Let

$x \in X$ such that $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = F_\alpha$. Again by definition of $\bigsqcup_\alpha$ it holds that $B(x) = F_\alpha$. Thus it holds that $\mathcal{S} \sqsubseteq_\alpha \bigsqcup_\alpha \mathcal{S}$.

Let $B \in (A]_\alpha$ such that $\mathcal{S} \sqsubseteq_\alpha B$. We prove that $\bigsqcup_\alpha \mathcal{S} \mid\mid T_\alpha \subseteq B \mid\mid T_\alpha$ and $\bigsqcup_\alpha \mathcal{S} \mid\mid F_\alpha \supseteq B \mid\mid F_\alpha$. Let $x \in X$ such that $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = T_\alpha$. Then by the definition of $\bigsqcup_\alpha$ there exists a $C \in \mathcal{S}$ such that $C(x) = T_\alpha$. Since $\mathcal{S} \sqsubseteq_\alpha B$, we have that $C \sqsubseteq_\alpha B$ and thus $B(x) = T_\alpha$. Let $x \in X$ such that $B(x) = F_\alpha$. Since $\mathcal{S} \sqsubseteq_\alpha B$ it is implied that $(\forall C \in \mathcal{S})\ C(x) = F_\alpha$. So by definition of $\bigsqcup_\alpha$, it holds that $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = F_\alpha$. So $\bigsqcup_\alpha \mathcal{S} \sqsubseteq_\alpha B$.

It remains to show that $\bigsqcup_\alpha \mathcal{S} \leq B$. Initially $\forall \beta < \alpha$ it holds that $\bigsqcup_\alpha \mathcal{S} =_\beta B$. Let $x \in X$ such that $order(B(x)) \geq \alpha$. We distinguish cases:

- If $B(x) = T_\alpha$ then by the definition of $\bigsqcup_\alpha$ either $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = T_\alpha$, $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = F_\alpha$ or $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = F_{\alpha+1}$. In any case $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) \leq B(x)$ holds.
- If $B(x) = F_\alpha$ then since $\bigsqcup_\alpha \mathcal{S} \sqsubseteq_\alpha B$ it holds that $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = F_\alpha$. Thus $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) \leq B(x)$.
- If $F_{\alpha+1} \leq B(x) \leq T_{\alpha+1}$ then since $\mathcal{S} \sqsubseteq_\alpha B$ it holds that $(\forall C \in \mathcal{S})\ F_\alpha \leq C(x) \leq T_{\alpha+1}$. If $(\forall C \in \mathcal{S})\ C(x) = F_\alpha$ then $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = F_\alpha$. If $(\exists C \in \mathcal{S})\ F_{\alpha+1} \leq C(x) \leq T_{\alpha+1}$ then by definition of $\bigsqcup_\alpha$, $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) = F_{\alpha+1}$. In either case $\big(\bigsqcup_\alpha \mathcal{S}\big)(x) \leq B(x)$.

In any case it holds that $\bigsqcup_\alpha \mathcal{S} \leq B$. □

The following theorem holds.

Theorem 2.20. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Let $(S_\alpha)_{\alpha \leq \kappa}$ be a sequence of lexicographic sets as in definition 2.18, defined on $\mathcal{S}$. Then S_κ is the least upper bound of $\mathcal{S}$ under the subset relation $\sqsubseteq$.

Proof. See proof of Theorem 4.2 in [9]. □

We will denote the least upper bound of $\mathcal{S}$ under $\sqsubseteq$ by $\bigsqcup \mathcal{S}$. Thus we define the (generalized) lexicographic union with respect to $\sqsubseteq$ as follows:

Definition 2.21. *Lexicographic union*. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. We define the lexicographic union of $\mathcal{S}$ as the least upper bound $\bigsqcup \mathcal{S}$ of $\mathcal{S}$ under $\sqsubseteq$.

Remark. As in the case of $\bigvee$, if $\mathcal{S} = \emptyset$ then $S_\kappa = S_0 = \bigvee_{\beta<0} S_\beta = \bigvee \emptyset$. As before $(\forall x \in X)\ \big(\bigvee \mathcal{S}\big)(x) = F_0$, since by definition of lub we have that $lub\ \emptyset = F_0$. Again this coincides with classical set theory.

Example 2.22. *Using κ, X and $\mathcal{S}$ as in example 2.17, the sequence $(S_\alpha)_{\alpha \leq \omega}$ is defined as follows. For each $\alpha < \omega$ we have $\mathcal{S}_\alpha = \{A_n \in \mathcal{S} \mid \alpha \leq n\}$ and $S_\alpha = \{\langle \alpha, F_{\alpha+1} \rangle\}$. Then $\bigsqcup \mathcal{S} = S_\omega = \bigvee_{\alpha<\omega} S_\alpha = \{\langle a, lub\{F_{\alpha+1} \mid \alpha < \omega\} \rangle\} = \{\langle a, 0 \rangle\}$.*

2.3.3 Relationship between the two

In this subsection we determine the relationship between $\bigvee \mathcal{S}$ and $\bigsqcup \mathcal{S}$ for a given set of lexicographic sets $\mathcal{S}$. We prove the following lemmas.

Lemma 2.23. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\bigsqcup \mathcal{S} \leq \bigvee \mathcal{S}$$

Proof. Let $(S_\alpha)_{\alpha\leq\kappa}$ be the sequence of lexicographic sets such that $\bigsqcup\mathcal{S} = S_\kappa = \bigvee_{\alpha<\kappa} S_\alpha$. We prove that $(\forall\alpha < \kappa)\ S_\alpha \leq \bigvee\mathcal{S}$. For $\alpha = 0$ our claim holds (the proof is similar to the induction step). Let $0 < \alpha < \kappa$ be an ordinal such that $(\forall\beta < \alpha)\ S_\beta \leq \bigvee\mathcal{S}$. We distinguish cases:

- $S_\alpha = \bigsqcup_\alpha \mathcal{S}_\alpha$. Initially it holds that $(\forall\beta < \alpha)\ S_\beta =_\beta S_\alpha$. Thus let $\beta = order(S_\alpha(x))$ for any $x \in X$ such that $order(S_\alpha(x)) < \alpha$. Then $S_\alpha(x) = S_\beta(x) \leq \big(\bigvee\mathcal{S}\big)(x)$ which holds by the hypothesis. Now let $x \in X$ such that $order(S_\alpha(x)) \geq \alpha$. We distinguish cases:
  - $S_\alpha(x) = F_\alpha$. Then by definition of $\bigsqcup_\alpha$, it holds that $(\forall A \in \mathcal{S}_\alpha)\ A(x) = F_\alpha$. It also holds that $\mathcal{S}_\alpha \subseteq \mathcal{S}$ and thus $(\forall A \in \mathcal{S}_\alpha)\ A \leq \bigvee\mathcal{S}$. So $S_\alpha(x) \leq \big(\bigvee\mathcal{S}\big)(x)$ holds.
  - $S_\alpha(x) = T_\alpha$. Then by definition of $\bigsqcup_\alpha$, it holds that $(\exists A \in \mathcal{S}_\alpha)\ A(x) = T_\alpha$. Again since $A \leq \bigvee\mathcal{S}$ it is implied that $S_\alpha(x) \leq \big(\bigvee\mathcal{S}\big)(x)$.
  - $S_\alpha(x) = F_{\alpha+1}$. Then by definition of $\bigsqcup_\alpha$, it holds that $(\forall A \in \mathcal{S}_\alpha)\ A(x) \neq T_\alpha$ and $(\exists A \in \mathcal{S}_\alpha)\ A(x) \neq F_\alpha$. Let $B \in \mathcal{S}_\alpha$ such that $B(x) \neq F_\alpha$. By definition of $\mathcal{S}_\alpha$ it holds that $(\forall\beta < \alpha)\ S_\alpha =_\beta S_\beta =_\beta B$ and thus $order(B(x)) \geq \alpha$. Since $B(x) \neq T_\alpha$ and $B(x) \neq F_\alpha$ it is implied that $F_{\alpha+1} \leq B(x) \leq T_{\alpha+1}$. Since $B \leq \bigvee\mathcal{S}$ it holds that $S_\alpha(x) \leq \big(\bigvee\mathcal{S}\big)(x)$.
- $S_\alpha = \bigvee_{\beta<\alpha} S_\beta$. By the hypothesis it holds that $(\forall\beta < \alpha)\ S_\beta \leq \bigvee\mathcal{S}$. By the definition of $\bigvee$ it holds that $S_\alpha \leq \bigvee\mathcal{S}$.

In any case it holds that $S_\alpha \leq \bigvee\mathcal{S}$. Thus it holds that $(\forall\alpha < \kappa)\ S_\alpha \leq \bigvee\mathcal{S}$ and again by the definition of $\bigvee$ it is implied that $\bigsqcup\mathcal{S} = S_\kappa \leq \bigvee\mathcal{S}$. □

Lemma 2.24. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\bigsqcup\mathcal{S} \sqsubseteq \bigvee\mathcal{S}$$

Proof. By lemma 2.23, $\bigsqcup\mathcal{S} \leq \bigvee\mathcal{S}$. Then lemma 2.13 implies that $\bigsqcup\mathcal{S} \sqsubseteq \bigvee\mathcal{S}$. □

Remark. $\bigsqcup\mathcal{S} = \bigvee\mathcal{S}$ does not hold in general. For example, let $X = \{a, b\}$ and $\mathcal{S} = \{A, B\}$, where $A = \{\langle a, T_1\rangle, \langle b, F_0\rangle\}$, $B = \{\langle a, F_0\rangle, \langle b, F_1\rangle\}$. Then $\bigvee\mathcal{S} = \{\langle a, T_1\rangle, \langle b, F_1\rangle\}$ while $\bigsqcup\mathcal{S} = \{\langle a, F_1\rangle, \langle b, F_1\rangle\}$.

In the next section we introduce the equivalent definitions regarding intersections of lexicographic sets.

2.4 Intersection operations

As in the case of the union operations we define one (generalized) intersection operation for each of the subset relations $\leq$ and $\sqsubseteq$. Following the union definitions we use the greatest lower bound under these relations, to define the intersection.

2.4.1 Point-wise intersection

We define the (generalized) point-wise intersection with respect to $\leq$ as follows:

Definition 2.25. *Point-wise intersection*. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. We define the point-wise intersection of $\mathcal{S}$ to be the greatest lower bound $\bigwedge \mathcal{S}$ of $\mathcal{S}$ under $\leq$. Let $x \in X$, then:

$$\left(\bigwedge \mathcal{S}\right)(x) = glb\{A(x) \mid A \in \mathcal{S}\}$$

Remark. If $\mathcal{S} = \emptyset$ then $(\forall x \in X)\ \big(\bigwedge \mathcal{S}\big)(x) = T_0$, since by definition of glb we have that $glb\ \emptyset = T_0$. As we will mention in a later section this is the $\leq$-maximum element in $\mathcal{X}$. This coincides with classical set theory where $\bigcap \emptyset$ represents the class containing all sets.

Example 2.26. *Let $\kappa = \omega$, let $X = \{a\}$ and $\mathcal{S} = \{A_n \mid n < \omega\}$, where $A_n = \{\langle a, T_n\rangle\}$. Then $\bigwedge \mathcal{S} = \{\langle a, glb\{T_n \mid n < \omega\}\rangle\} = \{\langle a, 0\rangle\}$.*

2.4.2 Lexicographic intersection

In [9] it is proven that a model of the Axioms paired with the relation $\sqsubseteq$ is a complete lattice and thus the existence of the greatest lower bound under $\sqsubseteq$ for any subset of $\mathcal{X}$ is guaranteed. Nevertheless its definition is not explicitly stated. We define in a symmetrical way to the least upper bound, the greatest lower bound under $\sqsubseteq$. We also prove that our definition indeed coincides with the greatest lower bound.

Let $A \in \mathcal{X}$ be a lexicographic set. Let $\mathcal{S} \subseteq (A]_\alpha$ for a given ordinal $\alpha < \kappa$. Then we define $\bigsqcap_\alpha S$ as follows. Let $x \in X$, then:

$$\left(\bigsqcap_\alpha \mathcal{S}\right)(x) = \begin{cases} A(x), & \text{If } order(A(x)) < \alpha \\ F_\alpha, & \text{If } (\exists B \in \mathcal{S})\ B(x) = F_\alpha \\ T_\alpha, & \text{If } (\forall B \in \mathcal{S})\ B(x) = T_\alpha \\ T_{\alpha+1}, & \text{otherwise} \end{cases}$$

As with the union, we define the following sequence of lexicographic sets.

Definition 2.27. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. We define a sequence of lexicographic sets $(S_\alpha)_{\alpha \leq \kappa}$ as follows. For all $\alpha < k$ we define:

$$\mathcal{S}_\alpha = \{A \in \mathcal{S} \mid (\forall \beta < \alpha)\ A =_\beta S_\beta\}$$

$$S_\alpha = \begin{cases} \bigsqcap_\alpha \mathcal{S}_\alpha, & \text{If } \mathcal{S}_\alpha \neq \emptyset \\ \bigwedge_{\beta<\alpha} S_\beta, & \text{If } \mathcal{S}_\alpha = \emptyset \end{cases}$$

Then:

$$S_\kappa = \bigwedge_{\alpha<\kappa} S_\alpha$$

Intuitively, as in the case of the definition 2.18, the above definition defines an infinite number of approximations of the greatest lower bound of the lexicographic sets in $\mathcal{S}$. As it proceeds through the ordinals $\alpha < \kappa$, each approximation S_α builds upon the previous approximations, improving the approximation, by constructing the α-th level to be an upper bound of the remaining lexicographic sets in $\mathcal{S}_\alpha$, as tight as possible, with respect to the lexicographic subset relation $\sqsubseteq$.

Before we proceed with the proof that the definition 2.27 indeed gives us the greatest lower bound of $\mathcal{S}$ under $\sqsubseteq$, we first establish a few important lemmas.

Lemma 2.28. Let $A \in \mathcal{S}$ be a lexicographic set. Let $\mathcal{S} \subseteq (A]_\alpha$ for ordinal $\alpha < \kappa$. Then:

- $\bigsqcap_\alpha \mathcal{S} \sqsubseteq_\alpha \mathcal{S}$
- $\big(\forall B \in (A]_\alpha\big)\ B \sqsubseteq_\alpha \mathcal{S} \rightarrow \big(B \sqsubseteq_\alpha \bigsqcap_\alpha \mathcal{S}\ \wedge\ B \leq \bigsqcap_\alpha \mathcal{S}\big)$

Proof. Since $\mathcal{S} \subseteq (A]_\alpha$ it holds that $(\forall B \in \mathcal{S})(\forall C \in \mathcal{S})(\forall \beta < \alpha)\ B =_\beta C$. So by the definition of $\bigsqcap_\alpha$, it holds that $\bigsqcap_\alpha \mathcal{S} =_\beta B =_\beta C$. Then $\bigsqcap_\alpha \in (A]_\alpha$.

Let $B \in \mathcal{S}$. We prove that $\bigsqcap_\alpha \mathcal{S} \mid\mid T_\alpha \subseteq B \mid\mid T_\alpha$ and $\bigsqcap_\alpha \mathcal{S} \mid\mid F_\alpha \supseteq B \mid\mid F_\alpha$. Let $x \in X$ such that $\big(\bigsqcap_\alpha \mathcal{S}\big)(x) = T_\alpha$. By the definition of $\bigsqcap_\alpha$ it holds that $B(x) = T_\alpha$. Let $x \in X$ such that $B(x) = F_\alpha$. Again by the definition of $\bigsqcap_\alpha$ it holds that $\big(\bigsqcap_\alpha \mathcal{S}\big)(x) = F_\alpha$. Thus it holds that $\bigsqcap_\alpha \mathcal{S} \sqsubseteq_\alpha \mathcal{S}$.

Let $B \in (A]_\alpha$ such that $B \sqsubseteq_\alpha \mathcal{S}$. We prove that $B \mid\mid T_\alpha \subseteq \bigsqcap_\alpha \mathcal{S} \mid\mid T_\alpha$ and $B \mid\mid F_\alpha \supseteq \bigsqcap_\alpha \mathcal{S} \mid\mid F_\alpha$. Let $x \in X$ such that $B(x) = T_\alpha$. Then since $B \sqsubseteq_\alpha \mathcal{S}$ it is implied that $(\forall C \in \mathcal{S})\ C(x) = T_\alpha$. So by the definition of $\bigsqcap_\alpha$, it holds that $\big(\bigsqcap_\alpha \mathcal{S}\big)(x) = T_\alpha$. Let $x \in X$ such that $\big(\bigsqcap_\alpha \mathcal{S}\big)(x) = F_\alpha$. By the definition of $\bigsqcap_\alpha$, there exists a $C \in \mathcal{S}$ such that $C(x) = F_\alpha$. Since $B \sqsubseteq_\alpha C$ it must hold that $B(x) = F_\alpha$. So $B \sqsubseteq_\alpha \bigsqcap_\alpha \mathcal{S}$.

It remains to show that $B \leq \bigsqcap_\alpha \mathcal{S}$. Initially $\forall \beta < \alpha$ it holds that $B =_\beta \bigsqcap_\alpha \mathcal{S}$. Let $x \in X$ such that $order(B(x)) \geq \alpha$. We distinguish cases:

- If $B(x) = T_\alpha$ then since $B \sqsubseteq_\alpha \bigsqcap_\alpha \mathcal{S}$ we have that $\big(\bigsqcap_\alpha \mathcal{S}\big)(x) = T_\alpha$. Thus $B(x) \leq \big(\bigsqcap_\alpha \mathcal{S}\big)(x)$.
- If $B(x) = F_\alpha$ then by the definition of $\bigsqcap_\alpha$, either $\big(\bigsqcap_\alpha\big)(x) = F_\alpha$ or $\big(\bigsqcap_\alpha\big)(x) = T_\alpha$ or $\big(\bigsqcap_\alpha\big)(x) = T_{\alpha+1}$. In any case $B(x) \leq \big(\bigsqcap_\alpha \mathcal{S}\big)(x)$ holds.
- If $F_{\alpha+1} \leq B(x) \leq T_{\alpha+1}$ then since $B \sqsubseteq_\alpha \mathcal{S}$ it holds that $(\forall C \in \mathcal{S})\ F_{\alpha+1} \leq C(x) \leq T_\alpha$. If $(\forall C \in \mathcal{S})\ C(x) = T_\alpha$ then $\big(\bigsqcap_\alpha \mathcal{S}\big)(x) = T_\alpha$. If $(\exists C \in \mathcal{S})\ F_{\alpha+1} \leq C(x) \leq T_{\alpha+1}$ then by definition $\big(\bigsqcap_\alpha \mathcal{S}\big)(x) = T_{\alpha+1}$. In either case $B(x) \leq \big(\bigsqcap_\alpha \mathcal{S}\big)(x)$.

In any case it holds that $B \leq \bigsqcap_\alpha \mathcal{S}$. □

Now let $\alpha < \kappa$ be an ordinal and $A \in \mathcal{X}$ be a lexicographic set. We define:

$$[A]_\alpha = \{B \in \mathcal{X} \mid (\forall \beta \leq \alpha)\ A =_\beta B\}$$

Notice that $[A]_\alpha \subseteq (A]_\alpha$.

Lemma 2.29. Let $\alpha < \kappa$ be an ordinal. Let $A \in \mathcal{X}$ be a lexicographic set. Let $\mathcal{S} \subseteq (A]_\alpha$. Let $B = \bigsqcap_\alpha \mathcal{S}$. Then B is the $\leq$-maximum element of $[B]_\alpha$. Also B is $\sqsubseteq_{\alpha+1}$-maximum element of $[B]_\alpha$.

Proof. Let $C \in [B]_\alpha$. By definition of $[B]_\alpha$ it holds that $(\forall \beta \leq \alpha)\ C =_\beta A$. Let $x \in X$ such that $order(C(x)) > \alpha$. By definition of $\bigsqcap_\alpha \mathcal{S}$, $B(x) = T_{\alpha+1}$ and thus $C(x) \leq B(x)$. So $C \leq B$ holds and B is the $\leq$-maximum element of $[B]_\alpha$.

We prove now that $C \mid\mid T_{\alpha+1} \subseteq B \mid\mid T_{\alpha+1}$ and $C \mid\mid F_{\alpha+1} \supseteq B \mid\mid F_{\alpha+1}$. Let $x \in X$ such that $C(x) = T_{\alpha+1}$. Since $order(C(x)) > \alpha$ it is immediately implied that $B(x) = T_{\alpha+1}$. In addition, by definition of $\bigsqcap_\alpha \mathcal{S}$ it holds that $B \mid\mid F_{\alpha+1} = \emptyset$, so the second inclusion holds trivially. Thus $C \sqsubseteq_{\alpha+1} B$ holds and B is $\sqsubseteq_{\alpha+1}$-maximum in $[B]_\alpha$. □

Lemma 2.30. Let $(A_\beta)_{\beta<\alpha}$ where $\alpha \leq \kappa$, be a sequence of lexicographic sets such that $A_\gamma =_\gamma A_\beta$ and $A_\beta \leq A_\gamma$ for $\gamma < \beta < \alpha$. Also let $A = \bigwedge_{\beta<\alpha} A_\beta$. Then $(\forall\beta < \alpha)\ A =_\beta A_\beta$.

Proof. If $\alpha = 0$ our claim holds trivially. Let $0 < \beta < \alpha$ be an ordinal such that $(\forall\gamma < \beta)\ A =_\gamma A_\gamma =_\gamma A_\beta$. We distinguish two cases.

- $A_\beta\#\beta = \emptyset$. Then for any δ such that $\beta < \delta$ by the hypothesis we have that $A_\delta\#\beta = \emptyset$. Also consider any δ such that $\delta < \beta$ with $A_\delta\#\beta \neq \emptyset$. Let $x \in X$. Observe that $order(A(x)) \neq \beta$ since $A_\beta \leq A_\delta$ and since $A_\beta\#\beta = \emptyset$ implies $order(A_\beta(x)) \neq \beta$. Thus $A\#\beta = \emptyset$.

- $A_\beta\#\beta \neq \emptyset$. Let $x \in X$ such that $order(A_\beta(x)) = \beta$. For any δ such that $\delta < \beta$ we have that $A_\beta(x) \leq A_\delta(x)$ by the hypothesis. Also for any δ such that $\beta < \delta$ we have that $A_\beta(x) = A_\delta(x)$ again by the hypothesis. Thus $A(x) = A_\beta(x)$ and $A\#\beta = A_\beta\#\beta$.

In any case $A\#\beta = A_\beta\#\beta$. Then it holds that $(\forall\beta < \alpha)\ A =_\beta A_\beta$. □

Lemma 2.31. Let $(A_\beta)_{\beta<\alpha}$ where $\alpha \leq \kappa$, be a sequence of lexicographic sets such that $A_\beta =_\beta A_\gamma$ for $\beta < \gamma$ and $\forall\beta < \alpha\ A_\beta$ is the $\leq$-maximum and a $\sqsubseteq_{\beta+1}$-maximum element of $[A_\beta]_\beta$. Let $B \in \mathcal{X}$ be a lexicographic set such that $(\forall\beta < \alpha)\ B \sqsubseteq_\beta A_\beta$. Also let $A = \bigwedge_{\beta<\alpha} A_\beta$. Then $B \sqsubseteq A$.

Proof. We first show that A is the $\leq$-maximum element and a $\sqsubseteq_\alpha$-maximum in $(A]_\alpha$. Since for all $\beta < \gamma < \alpha$ we have that $A_\beta =_\beta A_\gamma$ it holds that $[A_\gamma]_\gamma \subseteq [A_\beta]_\beta$. Also since A_β is the $\leq$-maximum element of $[A_\beta]_\beta$ and $[A_\gamma]_\gamma \subseteq [A_\beta]_\beta$, $A_\gamma \leq A_\beta$ holds. Using lemma 2.30 we have that $(\forall\beta < \alpha)\ A =_\beta A_\beta$. Trivially $A \in (A]_\alpha$ holds. Now let $C \in (A]_\alpha$. By definition we have that $A =_\beta A_\beta =_\beta C$ for any $\beta < \alpha$. Thus $(\forall\beta < \alpha)\ C \in [A_\beta]_\beta$ and since A_β is the $\leq$-maximum element of $[A_\beta]_\beta$, $C \leq A_\beta$ holds. By the definition of $\bigwedge$, it is implied that $C \leq A$. By Lemma 2.13 it holds that $C \sqsubseteq A$. This implies that $C \sqsubseteq_\alpha A$ since $(\forall\beta < \alpha)\ C =_\beta A$. Thus A is the $\leq$-maximum and a $\sqsubseteq_\alpha$-maximum in $(A]_\alpha$.

To complete the proof, we distinguish two cases. First, suppose that $\exists\beta < \alpha$ such that $B \sqsubset_\beta A_\beta$. Then since $A =_\beta A_\beta$ it is implied that $B \sqsubset_\beta A$, so $B \sqsubseteq A$ holds. Next, suppose that $(\forall\beta < \alpha)\ B =_\beta A_\beta$. Then $B \in (A]_\alpha$. Since A is the $\leq$-maximum element of $(A]_\alpha$, $B \leq A$ holds and Lemma 2.13 implies that $B \sqsubseteq A$. □

We can now prove the following theorem.

Theorem 2.32. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Let $(S_\alpha)_{\alpha\leq\kappa}$ be a sequence of lexicographic sets as in definition 2.27, defined on $\mathcal{S}$. Then S_κ is the greatest lower bound of $\mathcal{S}$ under the subset relation $\sqsubseteq$.

Proof. Initially we prove that $S_\beta =_\beta S_\alpha$, for all $\beta < \alpha < \kappa$ by induction on α. Our claim holds trivially for $\alpha = 0$. Let $0 < \alpha < \kappa$ and suppose that our claim holds for all $\beta < \alpha$. Let $\beta < \alpha$ be an ordinal. We distinguish two cases:

- $\mathcal{S}_\alpha \neq \emptyset$. Then $S_\alpha = \bigsqcap_\alpha \mathcal{S}_\alpha$. Let $A \in \mathcal{S}_\alpha$. By definition of $\mathcal{S}_\alpha$, $A =_\beta S_\beta$. Also by Lemma 2.28, $S_\alpha \sqsubseteq_\alpha A$. This implies that $S_\alpha =_\beta A$ and thus $S_\beta =_\beta S_\alpha$.

- $\mathcal{S}_\alpha = \emptyset$. Observe that for any ordinals $\gamma < \delta < \alpha$ the definitions of $\mathcal{S}_\gamma$ and $\mathcal{S}_\delta$ of definition 2.27 imply that $\mathcal{S}_\delta \subseteq \mathcal{S}_\gamma$. Also by the induction hypothesis we have that $S_\gamma =_\gamma S_\delta$ and thus $[S_\delta]_\delta \subseteq [S_\gamma]_\gamma$. By lemma 2.29 we have that S_γ is the $\leq$-maximum element of $[S_\gamma]_\gamma$ and since $S_\delta \in [S_\delta]_\delta$, $S_\delta \leq S_\gamma$. Thus we can assume that α is the least ordinal such that $\mathcal{S}_\alpha = \emptyset$ since for any $\gamma > \alpha$ it holds that $\mathcal{S}_\gamma = \emptyset$ and $S_\gamma = \bigwedge_{\delta<\gamma} S_\delta = S_\alpha$. Then we can apply lemma 2.30 which implies that $S_\beta =_\beta S_\alpha$.

From the previous induction we have that for any $\beta < \alpha < \kappa$, $S_\beta =_\beta S_\alpha$ and it also implies that $S_\alpha \leq S_\beta$. Then lemma 2.30 also implies that $(\forall \alpha < \kappa)\ S_\alpha =_\alpha S_\kappa$.

Let $\mathcal{S}_\kappa = \bigcap_{\alpha<\kappa} \mathcal{S}_\alpha$. We now proceed to show that either $\mathcal{S}_\kappa = \emptyset$ or $\mathcal{S}_\kappa = \{S_\kappa\}$. Suppose that $A \in \mathcal{S}_\kappa$. Then by definition of $\mathcal{S}_\kappa$ we have that $(\forall \alpha < \kappa)\ A =_\alpha S_\alpha =_\alpha S_\kappa$, which immediately implies that $A = S_\kappa$.

Now we prove that $S_\kappa \sqsubseteq \mathcal{S}$. Let $A \in \mathcal{S}$. Either $A \in \mathcal{S}_\kappa$ or $A \notin \mathcal{S}_\kappa$. If $A \in \mathcal{S}_\kappa$ then $A = S_\kappa$ and $S_\kappa \sqsubseteq A$ since $\sqsubseteq$ is reflexive. If $A \notin \mathcal{S}_\kappa$, let $\alpha < \kappa$ be the largest ordinal such that $A \in \mathcal{S}_\alpha$. Then $S_\alpha = \bigsqcap_\alpha \mathcal{S}_\alpha$ and by lemma 2.28 $S_\alpha \sqsubseteq_\alpha A$. Since $A \notin \mathcal{S}_{\alpha+1}$, by definition of $\mathcal{S}_{\alpha+1}$, $S_\alpha \neq_\alpha A$ and thus $S_\alpha \sqsubset_\alpha A$. Since $S_\alpha =_\alpha S_\kappa$ it holds that $S_\kappa \sqsubseteq A$.

Now consider any lower bound of $\mathcal{S}$, i.e. any $L \in \mathcal{X}$ such that $L \sqsubseteq \mathcal{S}$. We prove that $L \sqsubseteq S_\kappa$. If $\mathcal{S}_\kappa \neq \emptyset$ then our claim holds trivially since $S_\kappa \in \mathcal{S}$. Suppose that $\mathcal{S}_\kappa = \emptyset$. Let $\alpha < \kappa$ be the least ordinal such that $\mathcal{S}_\alpha = \emptyset$. If such an ordinal does not exist then let $\alpha = \kappa$. We show by induction on $\beta < \alpha$ that either $L \sqsubseteq_\beta S_\beta$ or $(\exists \gamma < \beta)\ L \sqsubset_\gamma S_\gamma$.

- For $\beta = 0$, we have that $S_0 = \bigsqcap_0 \mathcal{S}$. Since $L \sqsubseteq \mathcal{S}$ by definition of $\sqsubseteq$, $L \sqsubseteq_0 \mathcal{S}$ also holds. Then lemma 2.28 implies that $L \sqsubseteq_0 S_0$.
- Let $\beta > 0$ and suppose our claim holds for any ordinal $< \beta$. If there exists $\gamma < \beta$ such that $L \sqsubset_\gamma S_\gamma$ we are done. Otherwise the induction hypothesis implies that $(\forall \gamma < \beta)\ L \sqsubseteq_\gamma S_\gamma$. Since $(\forall \gamma < \beta)\ L \not\sqsubset_\gamma S_\gamma$ it holds that $(\forall \gamma < \beta)\ L =_\gamma S_\gamma =_\gamma S_\beta$. Since $\beta < \alpha$ we have that $\mathcal{S}_\beta \neq \emptyset$ and $S_\beta = \bigsqcap_\beta \mathcal{S}_\beta$. Since $L \sqsubseteq \mathcal{S}$ and $\mathcal{S}_\beta \subseteq \mathcal{S}$ it holds that $L \sqsubseteq \mathcal{S}_\beta$. Now let $A \in \mathcal{S}_\beta$. By the definition of $\mathcal{S}_\beta$, we have that for all $\gamma < \beta$, $A =_\gamma S_\gamma$. Since $S_\gamma =_\gamma S_\beta$ it is implied that $A =_\gamma S_\beta$. Then since $L =_\gamma S_\beta$, we have that $L =_\gamma A$ and thus since $L \sqsubseteq \mathcal{S}_\beta$ it is implied that $L \sqsubseteq_\beta \mathcal{S}_\beta$. Finally lemma 2.28 implies that $L \sqsubseteq_\beta S_\beta$.

To conclude the proof we observe the following. If $(\exists \gamma < \alpha)\ L \sqsubset_\gamma S_\gamma$ then since $S_\gamma =_\gamma S_\kappa$ we have that $L \sqsubseteq S_\kappa$. Otherwise our previous claim suggests that $(\forall \beta < \alpha)\ L \sqsubseteq_\beta S_\beta$ where $S_\alpha = \bigwedge_{\beta<\alpha} S_\beta$. We have previously proved that for any $\gamma < \beta < \alpha$, $S_\gamma =_\gamma S_\beta$ and $S_\beta \leq S_\gamma$ hold. By lemma 2.31 we have that $L \sqsubseteq S_\alpha$. Since $\mathcal{S}_\alpha = \emptyset$ it holds that $S_\alpha = S_\kappa$ and thus $L \sqsubseteq S_\kappa$. ☐

We will denote the greatest lower bound of $\mathcal{S}$ under $\sqsubseteq$ by $\bigsqcap \mathcal{S}$. Thus we define the (generalized) lexicographic intersection with respect to $\sqsubseteq$ as follows:

Definition 2.33. *Lexicographic intersection*. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. We define the lexicographic intersection of $\mathcal{S}$ to be the greatest lower bound $\bigsqcap \mathcal{S}$ of $\mathcal{S}$ under $\sqsubseteq$.

Remark. As in the case of $\bigwedge$, if $\mathcal{S} = \emptyset$ then $S_\kappa = S_0 = \bigwedge_{\beta<0} S_\beta = \bigwedge \emptyset$. As before $(\forall x \in X)\ \big(\bigvee \mathcal{S}\big)(x) = T_0$, since by definition of *glb* we have that *glb* $\emptyset = F_0$. Again this coincides with classical set theory.

Example 2.34. *Using κ, X and $\mathcal{S}$ as in example 2.26, the sequence $(S_\alpha)_{\alpha\leq\omega}$ is defined as follows. For each $\alpha < \omega$ we have $\mathcal{S}_\alpha = \{A_n \in \mathcal{S} \mid \alpha \leq n\}$ and $S_\alpha = \{\langle \alpha, F_{\alpha+1}\rangle\}$. Then $\bigsqcup \mathcal{S} = S_\omega = \bigvee_{\alpha<\omega} S_\alpha = \{\langle a, lub\{T_{\alpha+1} \mid \alpha < \omega\}\rangle\} = \{\langle a, 0\rangle\}$.*

2.4.3 Relationship between the two

In this subsection we determine the relationship between $\bigwedge \mathcal{S}$ and $\bigsqcap \mathcal{S}$ for a given set of lexicographic sets $\mathcal{S}$, in the same sense as in the previous section. We prove the equivalent lemmas.

Lemma 2.35. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\bigwedge \mathcal{S} \leq \bigsqcap \mathcal{S}$$

Proof. Let $(S_\alpha)_{\alpha \leq \kappa}$ be the sequence of lexicographic sets such that $\bigsqcap \mathcal{S} = S_\kappa = \bigwedge_{\alpha<\kappa} S_\alpha$. We prove that $(\forall \alpha < \kappa)\ \bigwedge \mathcal{S} \leq S_\alpha$. For $\alpha = 0$ our claim holds (the proof is similar to the induction step). Let $0 < \alpha < \kappa$ be an ordinal such that $(\forall \beta < \alpha)\ \bigwedge \mathcal{S} \leq S_\beta$. We distinguish cases:

- $S_\alpha = \bigsqcap_\alpha \mathcal{S}_\alpha$. Initially it holds that $(\forall \beta < \alpha)\ S_\beta =_\beta S_\alpha$. Thus let $\beta = order(S_\alpha(x))$ for any $x \in X$ such that $order(S_\alpha(x)) < \alpha$. Then $\big(\bigwedge \mathcal{S}\big)(x) \leq S_\beta(x) = S_\alpha(x)$ which holds by the hypothesis. Now let $x \in X$ such that $order(S_\alpha(x)) \geq \alpha$. We distinguish cases:
  - $S_\alpha(x) = T_\alpha$. Then by definition of $\bigsqcap_\alpha$, it holds that $(\forall A \in \mathcal{S}_\alpha)\ A(x) = T_\alpha$. It also holds that $\mathcal{S}_\alpha \subseteq \mathcal{S}$ and thus $(\forall A \in \mathcal{S}_\alpha)\ \bigwedge \mathcal{S} \leq A$. So $\big(\bigvee \mathcal{S}\big)(x) \leq S_\alpha(x)$ holds.
  - $S_\alpha(x) = F_\alpha$. Then by definition of $\bigsqcap_\alpha$, it holds that $(\exists A \in \mathcal{S}_\alpha)\ A(x) = F_\alpha$. Again since $\bigwedge \mathcal{S} \leq A$ it is implied that $\big(\bigvee \mathcal{S}\big)(x) \leq S_\alpha(x)$.
  - $S_\alpha(x) = T_{\alpha+1}$. Then by definition of $\bigsqcap_\alpha$, it holds that $(\forall A \in \mathcal{S}_\alpha)\ A(x) \neq F_\alpha$ and $(\exists A \in \mathcal{S}_\alpha)\ A(x) \neq T_\alpha$. Let $B \in \mathcal{S}_\alpha$ such that $B(x) \neq T_\alpha$. By definition of $\mathcal{S}_\alpha$ it holds that $(\forall \beta < \alpha)\ B =_\beta S_\beta =_\beta S_\alpha$ and thus $order(B(x)) \geq \alpha$. Since $B(x) \neq T_\alpha$ and $B(x) \neq F_\alpha$ it is implied that $F_{\alpha+1} \leq B(x) \leq T_{\alpha+1}$. Since $\bigwedge \mathcal{S} \leq B$ it holds that $\big(\bigwedge \mathcal{S}\big)(x) \leq S_\alpha(x)$.
- $S_\alpha = \bigwedge_{\beta<\alpha} S_\beta$. By the hypothesis it holds that $(\forall \beta < \alpha)\ \bigwedge \mathcal{S} \leq S_\beta$. By the definition of $\bigwedge$ it holds that $\bigwedge \mathcal{S} \leq S_\alpha$.

In any case it holds that $\bigwedge \mathcal{S} \leq S_\alpha$. Thus it holds that $(\forall \alpha < \kappa)\ \bigwedge \mathcal{S} \leq S_\alpha$ and again by the definition of $\bigwedge$ it is implied that $\bigvee \mathcal{S} \leq S_\kappa = \bigsqcap \mathcal{S}$. □

Lemma 2.36. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\bigwedge \mathcal{S} \sqsubseteq \bigsqcap \mathcal{S}$$

Proof. By lemma 2.35, $\bigwedge \mathcal{S} \leq \bigsqcap \mathcal{S}$. Then lemma 2.13 implies that $\bigwedge \mathcal{S} \sqsubseteq \bigsqcap \mathcal{S}$. □

Remark. Just as in the case of the union, $\bigwedge \mathcal{S} = \bigsqcap \mathcal{S}$ does not hold in general. For example, let $X = \{a, b\}$ and $\mathcal{S} = \{A, B\}$, where $A = \{\langle a, F_1\rangle, \langle b, T_0\rangle\}$, $B = \{\langle a, T_0\rangle, \langle b, T_1\rangle\}$. Then $\bigwedge \mathcal{S} = \{\langle a, F_1\rangle, \langle b, T_1\rangle\}$ while $\bigsqcap \mathcal{S} = \{\langle a, T_1\rangle, \langle b, T_1\rangle\}$.

2.5 Relationship between the lexicographic union and intersection

We prove the following simple lemma.

Lemma 2.37. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\bigsqcap \mathcal{S} \sqsubseteq \bigsqcup \mathcal{S}$$

Proof. Let $A \in \mathcal{S}$. It holds that $\bigsqcap \mathcal{S} \sqsubseteq A$ while $A \sqsubseteq \bigsqcup \mathcal{S}$. By lemma 2.15, the relation $\sqsubseteq$ is a partial order. Thus $\sqsubseteq$ is transitive. By transitivity of $\sqsubseteq$, $\bigsqcap \mathcal{S} \sqsubseteq \bigsqcup \mathcal{S}$ holds. ☐

The above trivial property, was of course expected to hold.

In the next section we introduce a definition for the complement of a lexicographic set along with some properties.

2.6 Complement operation

We start with some basic definitions.

Definition 2.38. Let $A, B \in \mathcal{X}$ be lexicographic sets. Then:

- $A \vee B = \bigvee\{A, B\}$ and similarly $A \sqcup B = \bigsqcup\{A, B\}$.
- $A \wedge B = \bigwedge\{A, B\}$ and similarly $A \sqcap B = \bigsqcap\{A, B\}$.

We also define the following two lexicographic sets.

Definition 2.39. We define $\top$ (respectively $\bot$) as follows: Let $x \in X$. Then $\top(x) = T_0$ (respectively $\bot(x) = F_0$).

Lemma 2.40. $\top$ (respectively $\bot$) is the maximum (minimum) element with respect to the subset relation $\leq$.

Proof. Let $A \in \mathcal{X}$ be a lexicographic set. Let $x \in X$. By the ordering of the truth values in V, we have that $A(x) \leq T_0 = \top(x)$. Thus $A \leq \top$. Symmetrically, $\bot(x) = F_0 \leq A(x)$. Thus $\bot \leq A$. ☐

Lemma 2.41. $\top$ (respectively $\bot$) is the maximum (minimum) element with respect to the subset relation $\sqsubseteq$.

Proof. Let $A \in \mathcal{X}$ be a lexicographic set. By lemma 2.40, $A \leq \top$. Then lemma 2.13 implies that $A \sqsubseteq \top$. In the same way $\bot \sqsubseteq A$ also holds. ☐

2.6.1 Complement definition

Ideally we would want to introduce a definition for the complement operation that naturally generalizes common properties that hold in the classical case.

Let $A \in \mathcal{X}$ be a lexicographic set. We symbolize its complement with $\overline{A}$. Some basic properties include:

- $\overline{\bot} = \top$
- $\overline{\top} = \bot$
- $\overline{(\overline{A})} = A$

If we want these properties to hold then defining the complement is straightforward.

Definition 2.42. *Lexicographic complement*. Let $A \in \mathcal{X}$ be a lexicographic set. Let $x \in X$. Then:

$$\overline{A}(x) = \begin{cases} T_\alpha, & \text{if } A(x) = F_\alpha \\ F_\alpha, & \text{if } A(x) = T_\alpha \\ 0, & \text{if } A(x) = 0 \end{cases}$$

It is trivial to show that the above properties hold.

2.6.2 Properties with respect to the point-wise relation

In this subsection we determine whether two other basic properties hold. Namely,

- $A \vee \overline{A} = \top$
- $A \wedge \overline{A} = \bot$

with respect to the subset relation $\leq$.

In the context of lexicographic sets we can actually prove that the following strictly more general properties. Let $A \in \mathcal{X}$ be a lexicographic set. We define the lexicographic sets $A_\top$ and $A_\bot$ as follows. Let $x \in X$. Then:

$$A_\bot(x) = \begin{cases} 0, & \text{If } A(x) = 0 \\ F_{order(A(x))}, & \text{otherwise} \end{cases}$$

and

$$A_\top(x) = \begin{cases} 0, & \text{If } A(x) = 0 \\ T_{order(A(x))}, & \text{otherwise} \end{cases}$$

We now prove the following lemma.

Lemma 2.43. Let $A \in \mathcal{X}$ be a lexicographic set. Then $A \vee \overline{A} = A_\top$ and $A \wedge \overline{A} = A_\bot$.

Proof. Let $x \in X$ such that $order(A(x)) = \alpha < \kappa$. Either $A(x) = T_\alpha$ or $\overline{A}(x) = T_\alpha$. By definition $(A \vee \overline{A})(x) = lub\{A(x), \overline{A}(x)\} = T_\alpha$. Also either $A(x) = F_\alpha$ or $\overline{A}(x) = F_\alpha$. By definition $(A \wedge \overline{A})(x) = glb\{A(x), \overline{A}(x)\} = F_\alpha$. Let $x \in X$ such that $order(A(x)) = \kappa$. Then trivially $(A \vee \overline{A})(x) = 0$ and $(A \wedge \overline{A})(x) = 0$. $\square$

Remark. Note that the lexicographic set $A_\top$ may not coincide with $\top$ while $A_\bot$ may not coincide with $\bot$. This is a property that strictly generalizes the classical properties where $A \cup \overline{A} = U$ and $A \cap \overline{A} = \emptyset$ for a classical set A in some universe U.

In the context of lexicographic sets another classical property can be strictly generalized. In the classical case it holds that $\emptyset \subseteq \overline{\emptyset} = U$ for some universe U. For any other arbitrary set A, where $A \neq \emptyset$ and $A \neq U$, it holds that $A \not\subseteq \overline{A}$ and $\overline{A} \not\subseteq A$. Thus the property $A \subseteq \overline{A}$ holds only for a single set, namely the empty set. However the following lemma holds in the context of lexicographic sets.

Lemma 2.44. There exists a collection $\mathcal{A}$ of lexicographic sets with cardinality at least κ, such that for any $A \in \mathcal{A}$ it holds that $A \leq \overline{A}$.

Proof. Let $\alpha < \kappa$ be an ordinal. Then we define a lexicographic set A_α as follows. For any $x \in X$, $A_\alpha(x) = F_\alpha$. Now let $\mathcal{A} = \{A_\alpha \mid \alpha < \kappa\}$. Observe that $A_\alpha \leq \overline{A_\alpha}$ holds. Also trivially the cardinality of $\mathcal{A}$ is κ. $\square$

2.6.3 Properties with respect to the lexicographic relation

As in the previous subsection we determine whether the following properties hold,

- $A \sqcup \overline{A} = \top$
- $A \sqcap \overline{A} = \bot$

with respect to the subset relation $\sqsubseteq$.

As with the relation $\leq$, we can actually prove that the following strictly more general properties. Let $A \in \mathcal{X}$ be a lexicographic set. We define the lexicographic sets $A_\top$ and $A_\bot$ as follows.

Let $x \in X$ and $\mu \leq \kappa$ be the least ordinal such that $A\#\mu \neq \emptyset$. Then:

$$A_\top(x) = \begin{cases} T_\mu, & \text{If } order(A(x)) = \mu < \kappa \\ 0, & \text{If } order(A(x)) = \mu = \kappa \\ A(x), & \text{If } order(A(x)) > \mu \text{ and } A =_\mu A_\top \\ \overline{A}(x), & \text{If } order(A(x)) > \mu \text{ and } \overline{A} =_\mu A_\top \\ F_{\mu+1}, & \text{otherwise} \end{cases}$$

Symmetrically,

$$A_\bot(x) = \begin{cases} F_\mu, & \text{If } order(A(x)) = \mu < \kappa \\ 0, & \text{If } order(A(x)) = \mu = \kappa \\ A(x), & \text{If } order(A(x)) > \mu \text{ and } A =_\mu A_\bot \\ \overline{A}(x), & \text{If } order(A(x)) > \mu \text{ and } \overline{A} =_\mu A_\bot \\ T_{\mu+1}, & \text{otherwise} \end{cases}$$

We now prove the following lemma.

Lemma 2.45. Let $A \in \mathcal{X}$ be a lexicographic set. Then $A \sqcup \overline{A} = A_\top$ and $A \sqcap \overline{A} = A_\bot$.

Proof. We prove that $A \sqcup \overline{A} = A_\top$. Let $(S_\alpha)_{\alpha \leq \kappa}$ be the sequence of elements such that $S_\kappa = A \sqcup \overline{A}$. Initially we know that for any $\beta < \alpha$, $S_\beta =_\beta S_\alpha$ and $S_\beta \leq S_\alpha$. We prove that $A \sqcup \overline{A} = A_\top$ by showing that $(\forall \alpha < \kappa)\ S_\alpha =_\alpha A_\top$. We prove our claim by induction on $\alpha < \kappa$.

Let $\alpha < \mu < \kappa$ and suppose that for any $\beta < \alpha$, $S_\beta =_\beta A_\top$. By definition of $A_\top$ we have that $S_\beta =_\beta A$. By definition of μ we also have that $A =_\beta \overline{A}$. Thus the definition of $\mathcal{S}_\alpha$ implies that $S_\alpha = \{A, \overline{A}\}$. Then $S_\alpha = \bigsqcup_\alpha \mathcal{S}_\alpha$. Since $A =_\alpha \overline{A}$ the definition of $\bigsqcup_\alpha$ implies that $S_\alpha =_\alpha A$. Since $A =_\alpha A_\top$ we have that $S_\alpha =_\alpha A_\top$.

Let $\alpha = \mu < \kappa$ and suppose that for any $\beta < \alpha$, $S_\beta =_\beta A_\top$. By definition of $A_\top$ and μ, for any $\beta < \alpha$ we have that $S_\beta =_\beta A =_\beta \overline{A}$. Then by definition of $\mathcal{S}_\alpha$, $\mathcal{S}_\alpha = \{A, B\}$ and $S_\alpha = \bigsqcup_\alpha \mathcal{S}_\alpha$. It suffices to show that $S_\alpha\#\alpha = A_\top\#\alpha$. Observe that $A_\top(x) = T_\alpha$ iff $order(A(x)) = \alpha$ iff $A(x) = T_\alpha$ or $\overline{A}(x) = T_\alpha$ iff $(\exists B \in \mathcal{S}_\alpha)\ B(x) = T_\alpha$ iff $S_\alpha(x) = T_\alpha$. This is implied by the definition of $\bigsqcup_\alpha$. Thus $S_\alpha =_\alpha A_\top$.

Let $\mu < \alpha < \kappa$ and suppose that for any $\beta < \alpha$, $S_\beta =_\beta A_\top$. We distinguish cases:

- $\mathcal{S}_\alpha \neq \emptyset$. Then either $A \in \mathcal{S}_\alpha$ or $\overline{A} \in \mathcal{S}_\alpha$ but not both since $A \neq_\mu \overline{A}$. Also $S_\alpha = \bigsqcup_\alpha \mathcal{S}$. In the first case, by the definition of $\bigsqcup_\alpha$, trivially $S_\alpha =_\alpha A$. By the induction hypothesis $S_\mu =_\mu A_\top$ and since $S_\mu =_\mu A$ we have that $A_\top =_\mu A$. Then the definition of $A_\top$ implies that $A_\top =_\alpha A$. Similarly when $\overline{A} \in \mathcal{S}_\alpha$. Finally $S_\alpha =_\alpha A_\top$ holds.

- $\mathcal{S}_\alpha = \emptyset$. Then let $\beta = \mu + 1$. Observe that β is the least ordinal such that $\mathcal{S}_\beta = \emptyset$. This holds since by definition of μ and the complement, $A \neq_\mu \overline{A}$. This implies that $\mathcal{S}_\beta = \{A\}$ or $\mathcal{S}_\beta = \{\overline{A}\}$ or $\mathcal{S}_\beta = \emptyset$. Observe that if $\mathcal{S}_\beta$ is a singleton then $\mathcal{S}_\alpha = \mathcal{S}_\beta$. Thus $\mathcal{S}_\beta = \emptyset$. Since for any $\gamma < \mu$, $S_\gamma \leq S_\mu$ then $S_\alpha = \bigvee_{\beta<\alpha} S_\beta = S_\mu$. By the definition of $\bigsqcup_\mu$, the definition of $A_\top$ and the induction hypothesis, if $\alpha = \beta$ we have that $S_\alpha =_\alpha A_\top$. Otherwise we have that $S_\alpha \# \alpha = A_\top \# \alpha = \emptyset$. In any case $S_\alpha =_\alpha A_\top$ holds.

Thus $(\forall \alpha < \kappa)\ S_\alpha =_\alpha A_\top$. Since $(\forall \alpha < \kappa)\ S_\alpha =_\alpha S_\kappa$ we have that $A \sqcup \overline{A} = A_\top$. The proof for $A \sqcap \overline{A} = A_\perp$ is symmetrical and is omitted. □

Remark. Note that as in the case of the point-wise relation, the lexicographic set $A_\top$ may not coincide with $\top$ while $A_\perp$ may not coincide with $\perp$. This is a property that strictly generalizes the classical properties where $A \cup \overline{A} = U$ and $A \cap \overline{A} = \emptyset$ for a classical set A in some universe U.

As in the case of the point-wise relation, the following lemma also holds with respect to the lexicographic subset relation.

Lemma 2.46. There exists a collection $\mathcal{A}$ of lexicographic sets with cardinality at least κ, such that for any $A \in \mathcal{A}$ it holds that $A \sqsubseteq \overline{A}$.

Proof. Define $\mathcal{A}$ as in the proof of lemma 2.44. □

In the next section we prove equivalent laws for this context to De Morgan laws in classical set theory.

2.7 De Morgan Laws

In this section we prove equivalent laws to De Morgan laws for the two subset relations we have defined.

Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. For what follows in this section, $\overline{\mathcal{S}}$ is interpreted as $\overline{\mathcal{S}} = \{\overline{A} \mid A \in \mathcal{S}\}$.

2.7.1 De Morgan laws with respect to the point-wise relation

Let $\overline{F_\alpha} = T_\alpha$ and $\overline{T_\alpha} = F_\alpha$. Also if $W \subseteq V$ then $\overline{W} = \{\overline{v} \mid v \in W\}$. We first prove the following lemma.

Lemma 2.47. Let $W \subseteq V$ be a subset of the set of truth values. Then:

$$lub\, W = l \leftrightarrow glb\, \overline{W} = \bar{l}$$

Proof. We first show that k is an upper bound of W if and only if $\overline{k}$ is a lower bound of $\overline{W}$. Let $v \in W$. We know that $v \leq k$. We distinguish cases:

- Suppose that $k = F_\alpha$ for some $\alpha < \kappa$. Then $\overline{k} = T_\alpha$.
  - If $v = F_\beta$ for some $\beta \leq \alpha$. Then $\overline{v} = T_\beta$. By the ordering of truth values $\overline{k} \leq \overline{v}$.
- Suppose that $k = 0$. Then $\overline{k} = 0$.

  - If $v = F_\beta$ for some $\beta < \kappa$. Then $\overline{v} = T_\beta$. By the ordering of truth values $\overline{k} \leq \overline{v}$.
  - If $v = 0$. Then $\overline{v} = 0$ and trivially $\overline{k} \leq \overline{v}$.

- Suppose that $k = T_\alpha$ for some $\alpha < \kappa$. Then $\overline{k} = F_\alpha$.
  - If $v = T_\beta$ for some $\alpha \leq \beta < \kappa$. Then $\overline{v} = F_\beta$. By the ordering of truth values $\overline{k} \leq \overline{v}$.
  - If $v = 0$. Then $\overline{v} = 0$. By the ordering of truth values $\overline{k} \leq \overline{v}$.
  - If $v = F_\beta$ for arbitrary $\beta < \kappa$. Then $\overline{v} = T_\beta$. Trivially $\overline{k} \leq \overline{v}$.

Next we prove that $lub\, W = l \leftrightarrow glb\, \overline{W} = \overline{l}$. Let $l = lub\, W$. If $l \in W$ then trivially $\overline{l} \in \overline{W}$ and $\overline{l} = glb\, \overline{W}$.

Suppose now that $l \notin W$. From the previous claim we have that $\overline{l}$ is a lower bound of $\overline{W}$. Suppose to the contrary that $glb\, \overline{W} \neq \overline{l}$. Let $g = glb\, \overline{W}$. Then it must hold that $\overline{l} < g$. From the previous claim $\overline{g}$ must also be an upper bound of W. This is a contradiction since $l = lub\, W$ while $\overline{g} < l$.

For all the statements above, the other direction is symmetrical. □

We can now proceed with the main lemmas.

Lemma 2.48. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\overline{\bigvee \mathcal{S}} = \bigwedge \overline{\mathcal{S}}$$

Proof. Let $x \in X$. $\left(\overline{\bigvee \mathcal{S}}\right)(x) = T_\alpha$ if and only if $\left(\bigvee \mathcal{S}\right)(x) = F_\alpha$ if and only if $lub\{A(x) \mid A \in \mathcal{S}\} = F_\alpha$ if and only if (holds using lemma 2.47) $glb\{B(x) \mid B \in \overline{\mathcal{S}}\} = T_\alpha$ if and only if $\left(\bigwedge \overline{\mathcal{S}}\right)(x) = T_\alpha$. In exactly the same way, $\left(\overline{\bigvee \mathcal{S}}\right)(x) = F_\alpha$ if and only if $\left(\bigwedge \overline{\mathcal{S}}\right)(x) = F_\alpha$ and $\left(\overline{\bigvee \mathcal{S}}\right)(x) = 0$ if and only if $\left(\bigwedge \overline{\mathcal{S}}\right)(x) = 0$. □

Lemma 2.49. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\overline{\bigwedge \mathcal{S}} = \bigvee \overline{\mathcal{S}}$$

Proof. Consider the complement of $\bigvee \overline{\mathcal{S}}$. By lemma 2.48 $\overline{\bigvee \overline{\mathcal{S}}} = \bigwedge \mathcal{S}$. By the complement's definition $\overline{\bigwedge \mathcal{S}} = \bigvee \overline{\mathcal{S}}$ holds. □

2.7.2 De Morgan laws with respect to the lexicographic relation

In this subsection we prove the equivalent lemmas for the $\sqsubseteq$ subset relation.

Lemma 2.50. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\overline{\bigsqcup \mathcal{S}} = \bigsqcap \overline{\mathcal{S}}$$

Proof. Let $(S_\alpha)_{\alpha \leq \kappa}$ be the sequence of elements such that $\bigsqcup \mathcal{S} = S_\kappa = \bigvee_{\alpha<\kappa} S_\alpha$ and $(U_\alpha)_{\alpha \leq \kappa}$ be the sequence of elements such that $\bigsqcap \overline{\mathcal{S}} = U_\kappa = \bigwedge_{\alpha<\kappa} U_\alpha$.

First we show that $(\forall \alpha < \kappa)\ \overline{S_\alpha} = U_\alpha$. We prove the previous claim by induction on α.

Let $\alpha = 0$. We have that $\mathcal{S}_0 = \mathcal{S}$ and $S_0 = \bigsqcup_0 \mathcal{S}_0$. Also $\mathcal{U}_0 = \overline{\mathcal{S}}$ and $U_0 = \bigsqcap_0 \mathcal{U}_0$. By complement's definition $\mathcal{S}_0 = \overline{\mathcal{U}_0}$. Let $x \in X$. We have that $S_0(x) = T_0$ iff

$(\exists A \in \mathcal{S}_0)\ A(x) = T_0$ iff $(\exists B \in \mathcal{U}_0)\ B(x) = F_0$ iff $U_0(x) = F_0$. Also $S_0(x) = F_0$ iff $(\forall A \in \mathcal{S}_0)\ A(x) = F_0$ iff $(\forall B \in \mathcal{U}_0)\ B(x) = T_0$ iff $U_0(x) = T_0$. Finally $S_0(x) = F_{\alpha+1}$ iff $(\forall A \in \mathcal{S}_0)\ A(x) \neq T_0$ and $(\exists A \in \mathcal{S}_0)\ A(x) \neq F_0$ iff $(\forall B \in \mathcal{U}_0)\ B(x) \neq F_0$ and $(\exists B \in \mathcal{U}_0)\ B(x) \neq T_0$ iff $U_0(x) = T_{\alpha+1}$. Thus by complement's definition, $\overline{S_0} = U_0$ holds.

Let $0 < \alpha < \kappa$ and suppose that our claim holds for all $\beta < \alpha$. First we show that $\mathcal{S}_\alpha = \overline{\mathcal{U}_\alpha}$. We have that $A \in \mathcal{S}_\alpha$ iff $(\forall \beta < \alpha)\ A =_\beta S_\beta$ and $A \in \mathcal{S}$ iff (by the induction hypothesis) $(\forall \beta < \alpha)\ A =_\beta \overline{U_\beta}$ and $A \in \mathcal{S}$ iff (by complement's definition) $(\forall \beta < \alpha)\ \overline{A} =_\beta U_\beta$ and $\overline{A} \in \overline{\mathcal{S}}$ iff $\overline{A} \in \mathcal{U}_\alpha$. Thus $\mathcal{S}_\alpha = \overline{\mathcal{U}_\alpha}$. Now we show that $\overline{S_\alpha} = U_\alpha$. We distinguish cases:

- $\mathcal{S}_\alpha = \overline{\mathcal{U}_\alpha} \neq \emptyset$. Then $S_\alpha = \bigsqcup_\alpha \mathcal{S}_\alpha$ and $U_\alpha = \bigsqcap_\alpha \mathcal{U}_\alpha$. Since $S_\beta =_\beta S_\alpha$ and $U_\beta =_\beta U_\alpha$, by the induction hypothesis it holds that $\overline{S_\alpha} =_\beta U_\alpha$ for all $\beta < \alpha$. Then by the definition of $\bigsqcup_\alpha$ and $\bigsqcap_\alpha$ it suffices to show that $\overline{S_\alpha}\#\alpha = U_\alpha\#\alpha$ and $\overline{S_\alpha}\#(\alpha+1) = U_\alpha\#(\alpha+1)$. Since $\mathcal{S}_\alpha = \overline{\mathcal{U}_\alpha}$ the proof of the induction base applies directly.
- $\mathcal{S}_\alpha = \overline{\mathcal{U}_\alpha} = \emptyset$. Then $S_\alpha = \bigvee_{\beta<\alpha} S_\beta$ and $U_\alpha = \bigwedge_{\beta<\alpha} U_\beta$. Using lemma 2.48 and the induction hypothesis we have that $\overline{S_\alpha} = \overline{\bigvee_{\beta<\alpha} S_\beta} = \bigwedge_{\beta<\alpha} \overline{S_\beta} = \bigwedge_{\beta<\alpha} U_\beta = U_\alpha$.

Thus $(\forall \alpha < \kappa)\ \overline{x_\alpha} = y_\alpha$ holds. Finally using lemma 2.48 we derive that:

$$\overline{\bigsqcup \mathcal{S}} = \overline{\bigvee_{\alpha<\kappa} S_\alpha} = \bigwedge_{\alpha<\kappa} \overline{S_\alpha} = \bigwedge_{\alpha<\kappa} U_\alpha = \bigsqcap \overline{\mathcal{S}}$$

□

Lemma 2.51. Let $\mathcal{S} \subseteq \mathcal{X}$ be a set of lexicographic sets. Then:

$$\overline{\bigsqcap \mathcal{S}} = \bigsqcup \overline{\mathcal{S}}$$

Proof. Consider the complement of $\bigsqcup \overline{\mathcal{S}}$. By lemma 2.50 $\overline{\bigsqcup \overline{\mathcal{S}}} = \bigsqcap \mathcal{S}$. By the complement's definition $\overline{\bigsqcap \mathcal{S}} = \bigsqcup \overline{\mathcal{S}}$ holds. □

2.8 Binary union and intersection properties

In this section we examine whether more classical properties of the union and intersection of lexicographic sets hold. First we simplify the binary union and intersection definitions.

We prove the following lemmas.

Lemma 2.52. Let $A, B \in \mathcal{X}$ be two lexicographic sets. Let $x \in X$. Then:

$$(A \vee B)(x) = \max\{A(x), B(x)\}$$

Symmetrically,

$$(A \wedge B)(x) = \min\{A(x), B(x)\}$$

Proof. Directly implied by definition 2.38 and the definitions of $\bigvee$ (see definition 2.16) and $\bigwedge$ (see definition 2.25). □

Lemma 2.53. Let $A, B \in \mathcal{X}$ be two lexicographic sets. Let $\mu < \kappa$ be the least ordinal such that $A \neq_\mu B$. We define the lexicographic sets U and I as follows. If μ exists, let $x \in X$. Then:

$$U(x) = \begin{cases} A(x), & \text{If } order(A(x)) < \mu \\ T_\mu, & x \in A \mid\mid T_\mu \cup B \mid\mid T_\mu \\ F_\mu, & x \in A \mid\mid F_\mu \cap B \mid\mid F_\mu \\ A(x), & \text{If } order(A(x)) > \mu \text{ and } A =_\mu U \\ B(x), & \text{If } order(B(x)) > \mu \text{ and } B =_\mu U \\ F_{\mu+1}, & \text{otherwise} \end{cases}$$

and

$$I(x) = \begin{cases} A(x), & \text{If } order(A(x)) < \mu \\ T_\mu, & x \in A \mid\mid T_\mu \cap B \mid\mid T_\mu \\ F_\mu, & x \in A \mid\mid F_\mu \cup B \mid\mid F_\mu \\ A(x), & \text{If } order(A(x)) > \mu \text{ and } A =_\mu I \\ B(x), & \text{If } order(B(x)) > \mu \text{ and } B =_\mu I \\ T_{\mu+1}, & \text{otherwise} \end{cases}$$

Else If μ does not exist then $U = I = A = B$. Then $A \sqcup B = U$ and $A \sqcap B = I$.

Proof. First if μ does not exist then trivially $A \sqcup B = A \sqcap B = A = B = U = I$. Suppose that μ exists. Let $(S_\alpha)_{\alpha \leq \kappa}$ be a sequence of lexicographic sets as in definition 2.18 such that $S_\kappa = A \sqcup B$. We prove that $A \sqcup B = U$ by showing that $(\forall \alpha < \kappa)\ S_\alpha =_\alpha U$.

Initially we know that for any $\beta < \alpha$, $S_\beta =_\beta S_\alpha$ and $S_\beta \leq S_\alpha$. We prove our claim by induction on $\alpha < \kappa$.

Let $\alpha < \mu$ and suppose that for any $\beta < \alpha$, $S_\beta =_\beta U$. By definition of U we have that $S_\beta =_\beta A$. By definition of μ we also have that $A =_\beta B$. Thus the definition of $\mathcal{S}_\alpha$ implies that $\mathcal{S}_\alpha = \{A, B\}$. Then $S_\alpha = \bigsqcup_\alpha \mathcal{S}_\alpha$. Since $A =_\alpha B$ the definition of $\bigsqcup_\alpha$ implies that $S_\alpha =_\alpha A$. Since $A =_\alpha U$ we have that $S_\alpha =_\alpha U$.

Let $\alpha = \mu$ and suppose that for any $\beta < \alpha$, $S_\beta =_\beta U$. By definition of U and μ, for any $\beta < \alpha$ we have that $S_\beta =_\beta A =_\beta B$. Then by definition of $\mathcal{S}_\alpha$, $\mathcal{S}_\alpha = \{A, B\}$ and $S_\alpha = \bigsqcup_\alpha \mathcal{S}_\alpha$. It suffices to show that $S_\alpha \# \alpha = U \# \alpha$. Observe that $U(x) = T_\alpha$ iff $x \in A \mid\mid T_\alpha \cup B \mid\mid T_\alpha$ iff $(\exists B \in \mathcal{S}_\alpha)\ B(x) = T_\alpha$ iff $S_\alpha(x) = T_\alpha$. Also $U(x) = F_\alpha$ iff $x \in A \mid\mid F_\alpha \cap B \mid\mid T_\alpha$ iff $(\forall B \in \mathcal{S}_\alpha)\ B(x) = F_\alpha$ iff $S_\alpha(x) = F_\alpha$. Both are implied by the definition of $\bigsqcup_\alpha$. Thus $S_\alpha =_\alpha U$.

Let $\mu < \alpha < \kappa$ and suppose that for any $\beta < \alpha$, $S_\beta =_\beta U$. We distinguish cases:

- $\mathcal{S}_\alpha \neq \emptyset$. Then either $A \in \mathcal{S}_\alpha$ or $B \in \mathcal{S}_\alpha$ but not both since $A \neq_\mu B$. Also $S_\alpha = \bigsqcup_\alpha \mathcal{S}$. In the first case, by the definition of $\bigsqcup_\alpha$, trivially $S_\alpha =_\alpha A$. By the induction hypothesis $S_\mu =_\mu U$ and since $S_\mu =_\mu A$ we have that $U =_\mu A$. Then the definition of U implies that $U =_\alpha A$. Similarly when $B \in \mathcal{S}_\alpha$. Finally $S_\alpha =_\alpha U$ holds.

- $\mathcal{S}_\alpha = \emptyset$. Then let $\beta = \mu + 1$. Observe that β is the least ordinal such that $\mathcal{S}_\beta = \emptyset$. This holds since by definition of μ, $A \neq_\mu B$. This implies that $\mathcal{S}_\beta = \{A\}$ or $\mathcal{S}_\beta = \{B\}$ or $\mathcal{S}_\beta = \emptyset$. Observe that if $\mathcal{S}_\beta$ is a singleton then $\mathcal{S}_\alpha = \mathcal{S}_\beta$. Thus $\mathcal{S}_\beta = \emptyset$. Since for any $\gamma < \mu$, $S_\gamma \leq S_\mu$ then $S_\alpha = \bigvee_{\beta < \alpha} S_\beta = S_\mu$. By the definition of $\bigsqcup_\mu$, the definition of U and the induction hypothesis, if $\alpha = \beta$ we have that $S_\alpha =_\alpha U$. Otherwise we have that $S_\alpha \# \alpha = U \# \alpha = \emptyset$. In any case $S_\alpha =_\alpha U$ holds.

Thus $(\forall \alpha < \kappa)\ S_\alpha =_\alpha U$. Since $(\forall \alpha < \kappa)\ S_\alpha =_\alpha S_\kappa$ we have that $A \sqcup B = U$. The proof for $A \sqcap B = I$ is symmetrical and is omitted. □

We continue by studying some classical properties.

2.8.1 Properties of the point-wise definitions

Lemma 2.54. Let $A, B, C \in \mathcal{X}$ be lexicographic sets. Then:

$$A \vee (B \vee C) = (A \vee B) \vee C \quad \text{and} \quad A \wedge (B \wedge C) = (A \wedge B) \wedge C$$

Proof. Let $x \in X$. From lemma 2.52 we have that:

$$\begin{aligned}
(A \vee (B \vee C))(x) &= \max\{A(x), (B \vee C)(x)\} \\
&= \max\{A(x), \max\{B(x), C(x)\}\} \\
&= \max\{A(x), B(x), C(x)\} \\
&= \max\{\max\{A(x), B(x)\}, C(x)\} \\
&= \max\{(A \vee B)(x), C(x)\} \\
&= ((A \vee B) \vee C)(x)
\end{aligned}$$

The proof for $(A \wedge (B \wedge C))(x) = ((A \wedge B) \wedge C)(x)$ is symmetrical. Thus the desired properties hold. □

Lemma 2.55. Let $A, B \in \mathcal{X}$ be lexicographic sets. Then:

$$A \vee B = B \vee A \quad \text{and} \quad A \wedge B = B \wedge A$$

Proof. Directly implied by lemma 2.52. □

Lemma 2.56. Let $A, B, C \in \mathcal{X}$ be lexicographic sets. Then:

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C) \quad \text{and} \quad A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

Proof. Let $x \in X$. First observe that

$$\min\{B(x), C(x)\} \leq \max\{B(x), C(x)\}$$

holds and it in turn implies that

$$\max\{A(x), \min\{B(x), C(x)\}\} \leq \max\{A(x), \max\{B(x), C(x)\}\}$$

Then using lemma 2.52 we have that:

$$\begin{aligned}
((A \vee B) \wedge (A \vee C))(x) &= \min\{(A \vee B)(x), (A \vee C)(x)\} \\
&= \min\{\max\{A(x), B(x)\}, \max\{A(x), C(x)\}\} \\
&= \min\{\max\{A(x), \min\{B(x), C(x)\}\}, \\
&\quad \max\{A(x), \max\{B(x), C(x)\}\}\} \\
&= \max\{A(x), \min\{B(x), C(x)\}\} \\
&= \max\{A(x), (B \wedge C)(x)\} \\
&= (A \vee (B \wedge C))(x)
\end{aligned}$$

Thus $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ holds. We can prove the second identity in a symmetrical way. □

Lemma 2.57. Let $A \in \mathcal{X}$ be a lexicographic set. Then:

$$A \vee \bot = A \quad \text{and} \quad A \wedge \top = A$$

Proof. By lemma 2.40, $\bot$ is the minimum element with respect to $\leq$ and thus $\bot \leq A$. Since we have defined $\vee$ as the least upper bound with respect to $\leq$ it is implied that $A \vee \bot = A$.

Symmetrically, $\top$ is the maximum element with respect to $\leq$ and thus $A \leq \top$. Since we have defined $\wedge$ as the greatest lower bound with respect to $\leq$ it is implied that $A \wedge \top = A$. □

Lemma 2.58. Let $B \in \mathcal{X}$ be a lexicographic set. Then if, for all $A \in \mathcal{X}$, $A \vee B = A$, then $B = \bot$. Additionally if, for all $A \in \mathcal{X}$, $A \wedge B = A$, then $B = \top$.

Proof. Let $A \in \mathcal{X}$. Since $A \vee B = A$ ($A \wedge B = A$) and $\vee$ ($\wedge$) is the lub (glb) with respect to $\leq$ it is implied that $B \leq A$ ($A \leq B$). Then it is implied that for all $A \in \mathcal{X}$, $B \leq A$ ($A \leq B$). Then it holds that $B \leq \bot$ ($\top \leq B$). By lemma 2.40 it also holds that $\bot \leq B$ ($B \leq \top$). Since $\leq$ is antisymmetric it holds that $B = \bot$ ($B = \top$). □

Lemma 2.59. Let $A, B \in \mathcal{X}$ be lexicographic sets. If there exists a $C \in \mathcal{X}$ such that $A \vee B = C$ and $A \wedge B = \overline{C}$, then $B = \overline{A}$.

Proof. Let $x \in X$. Then using lemma 2.52 we have that $\max\{A(x), B(x)\} = C(x)$ and $\min\{A(x), B(x)\} = \overline{C}(x)$. This directly implies that $B(x) = \overline{A}(x)$. By the complement's definition we have that $B = \overline{A}$. □

Remark. Lemma 2.59 generalizes the classical property when $C = \top$ and $\overline{C} = \bot$.

Lemma 2.60. Let $A \in \mathcal{X}$ be a lexicographic set. Then:

$$A \vee A = A \quad \text{and} \quad A \wedge A = A$$

Proof. Directly implied by lemma 2.52. □

Lemma 2.61. Let $A \in \mathcal{X}$ be a lexicographic set. Then:

$$A \vee \top = \top \quad \text{and} \quad A \wedge \bot = \bot$$

Proof. By lemma 2.40, $\top$ is the maximum element with respect to $\leq$ and thus $A \leq \top$. Since we have defined $\vee$ as the least upper bound with respect to $\leq$ it is implied that $A \vee \top = \top$.

Symmetrically, $\bot$ is the minimum element with respect to $\leq$ and thus $\bot \leq A$. Since we have defined $\wedge$ as the greatest lower bound with respect to $\leq$ it is implied that $A \wedge \bot = \bot$. □

Lemma 2.62. Let $A, B \in \mathcal{X}$ be lexicographic sets. Then:

$$A \vee (A \wedge B) = A \quad \text{and} \quad A \wedge (A \vee B) = A$$

Proof. Let $x \in X$. By lemma 2.52 we have that $(A \vee (A \wedge B))(x) = \max\{A(x), (A \wedge B)(x)\} = \max\{A(x), \min\{A(x), B(x)\}\}$. Observe that whether $A(x) \leq B(x)$ or $B(x) \leq A(x)$, $(A \vee (A \wedge B))(x) = A(x)$. Thus $A \vee (A \wedge B) = A$. Symmetrically $A \wedge (A \vee B) = A$ holds as well. □

Lemma 2.63. Let $A, B \in \mathcal{X}$ be lexicographic sets. Then:

$$\overline{A \vee B} = \overline{A} \wedge \overline{B} \quad \text{and} \quad \overline{A \wedge B} = \overline{A} \vee \overline{B}$$

Proof. Directly implied by lemmas 2.48 and 2.49. $\square$

Lemma 2.64. Let $A, B \in \mathcal{X}$ be lexicographic sets. The following statements are equivalent:

1. $A \leq B$
2. $A \wedge B = A$
3. $A \vee B = B$

Proof. 1 implies 2. Assume that $A \leq B$. Then $A \wedge B = A$ is directly implied since we have defined $\wedge$ as the greatest lower bound with respect to $\leq$.

2 implies 3. Assume that $A \wedge B = A$. Then $A \vee B = (A \wedge B) \vee B = B$ by lemma 2.62.

3 implies 1. Assume that $A \vee B = B$. Then since $A \leq A \vee B$ we have that $A \leq B$. $\square$

2.8.2 Properties of the lexicographic definitions

Lemma 2.65. Let $A, B, C \in \mathcal{X}$ be lexicographic sets. If $A \sqsubseteq C$ and $B \sqsubseteq C$ then,

$$A \sqcup B \sqsubseteq C$$

Also if $C \sqsubseteq A$ and $C \sqsubseteq B$ then,

$$C \sqsubseteq A \sqcap B$$

Proof. Since we have defined $\sqcup$ as the least upper bound with respect to $\sqsubseteq$ it implied that, $A \sqcup B \sqsubseteq C$.

Symmetrically, since we have defined $\sqcap$ as the greatest lower bound with respect to $\sqsubseteq$, it is implied that $C \sqsubseteq A \sqcap B$. $\square$

Lemma 2.66. Let $A, B, C \in \mathcal{X}$ be lexicographic sets. Then:

$$A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C \quad \text{and} \quad A \sqcap (B \sqcap C) = (A \sqcap B) \sqcap C$$

Proof. We prove the first property. The proof for the other one is symmetrical.

First we trivially have that $A \sqsubseteq A \sqcup (B \sqcup C)$ and that $B \sqcup C \sqsubseteq A \sqcup (B \sqcup C)$. We also have that $B \sqsubseteq B \sqcup C$ and that $C \sqsubseteq B \sqcup C$. Since $\sqsubseteq$ is transitive, $B \sqsubseteq A \sqcup (B \sqcup C)$ and $C \sqsubseteq A \sqcup (B \sqcup C)$ hold. Using lemma 2.65 we get that $A \sqcup B \sqsubseteq A \sqcup (B \sqcup C)$ and with another iteration that $(A \sqcup B) \sqcup C \sqsubseteq A \sqcup (B \sqcup C)$.

Next we have that $A \sqcup B \sqsubseteq (A \sqcup B) \sqcup C$ and that $C \sqsubseteq (A \sqcup B) \sqcup C$. We also have that $A \sqsubseteq A \sqcup B$ and that $B \sqsubseteq A \sqcup B$. Since $\sqsubseteq$ is transitive, $A \sqsubseteq (A \sqcup B) \sqcup C$ and $B \sqsubseteq (A \sqcup B) \sqcup C$ hold. Using lemma 2.65 we get that $B \sqcup C \sqsubseteq (A \sqcup B) \sqcup C$ and with another iteration that $A \sqcup (B \sqcup C) \sqsubseteq (A \sqcup B) \sqcup C$.

Since $\sqsubseteq$ is antisymmetric it holds that $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$.

$\square$

Lemma 2.67. There exists a universe X and lexicographic sets $A, B, C \in \mathcal{X}$ such that:

$$A \sqcup (B \sqcap C) \neq (A \sqcup B) \sqcap (A \sqcup C) \quad \text{and} \quad \overline{A} \sqcap (\overline{B} \sqcup \overline{C}) \neq (\overline{A} \sqcap \overline{B}) \sqcup (\overline{A} \sqcap \overline{C})$$

Proof. Let $X = \{a, b, c, d, e\}$. Then let:

$$\begin{aligned} A &= \{\langle a, T_0\rangle, \langle b, T_0\rangle, \langle c, F_0\rangle, \langle d, F_0\rangle, \langle e, T_2\rangle\} \\ B &= \{\langle a, T_0\rangle, \langle b, F_0\rangle, \langle c, T_0\rangle, \langle d, F_0\rangle, \langle e, T_3\rangle\} \\ C &= \{\langle a, T_0\rangle, \langle b, F_0\rangle, \langle c, F_0\rangle, \langle d, T_0\rangle, \langle e, T_4\rangle\} \end{aligned}$$

Then:

$$\begin{aligned} B \sqcap C &= \{\langle a, T_0\rangle, \langle b, F_0\rangle, \langle c, F_0\rangle, \langle d, F_0\rangle, \langle e, T_1\rangle\} \\ A \sqcup (B \sqcap C) &= \{\langle a, T_0\rangle, \langle b, T_0\rangle, \langle c, F_0\rangle, \langle d, F_0\rangle, \langle e, T_2\rangle\} \end{aligned}$$

Also:

$$\begin{aligned} A \sqcup B &= \{\langle a, T_0\rangle, \langle b, T_0\rangle, \langle c, T_0\rangle, \langle d, F_0\rangle, \langle e, F_1\rangle\} \\ A \sqcup C &= \{\langle a, T_0\rangle, \langle b, T_0\rangle, \langle c, F_0\rangle, \langle d, T_0\rangle, \langle e, F_1\rangle\} \\ (A \sqcup B) \sqcap (A \sqcup C) &= \{\langle a, T_0\rangle, \langle b, T_0\rangle, \langle c, F_0\rangle, \langle d, F_0\rangle, \langle e, T_1\rangle\} \end{aligned}$$

Observe that $A \sqcup (B \sqcap C) \neq (A \sqcup B) \sqcap (A \sqcup C)$. Also by complement's definition it also holds that $\overline{A \sqcup (B \sqcap C)} \neq \overline{(A \sqcup B) \sqcap (A \sqcup C)}$. By lemma 2.76 we have that $\overline{A \sqcup (B \sqcap C)} = \overline{A} \sqcap (\overline{B \sqcap C}) = \overline{A} \sqcap (\overline{B} \sqcup \overline{C})$ and $\overline{(A \sqcup B) \sqcap (A \sqcup C)} = (\overline{A \sqcup B}) \sqcup (\overline{A \sqcup C}) = (\overline{A} \sqcap \overline{B}) \sqcup (\overline{A} \sqcap \overline{C})$ Thus the second claim also holds. ☐

Remark. Unfortunately lemma 2.67 implies that the classical distributive laws do not apply in general for the lexicographic union and intersection.

Nevertheless the following lemma does hold.

Lemma 2.68. Let $A, B, C \in \mathcal{X}$ be lexicographic sets. Then:

$$A \sqcup (B \sqcap C) \sqsubseteq (A \sqcup B) \sqcap (A \sqcup C) \quad \text{and} \quad (A \sqcap B) \sqcup (A \sqcap C) \sqsubseteq A \sqcap (B \sqcup C)$$

Proof. We first prove that $A \sqcup (B \sqcap C) \sqsubseteq (A \sqcup B) \sqcap (A \sqcup C)$. It holds that $A \sqsubseteq A \sqcup B, A \sqcup C$. Then lemma 2.65 implies that $A \sqsubseteq (A \sqcup B) \sqcap (A \sqcup C)$. Also $B \sqsubseteq A \sqcup B$ and $C \sqsubseteq A \sqcup C$. Since $B \sqcap C \sqsubseteq B, C$ by transitivity of $\sqsubseteq$, we have that $B \sqcap C \sqsubseteq A \sqcup B, A \sqcup C$. Then lemma 2.65 implies that $B \sqcap C \sqsubseteq (A \sqcup B) \sqcap (A \sqcup C)$. By applying lemma 2.65 once more, we get that $A \sqcup (B \sqcap C) \sqsubseteq (A \sqcup B) \sqcap (A \sqcup C)$.

In a similar fashion we prove that $(A \sqcap B) \sqcup (A \sqcap C) \sqsubseteq A \sqcap (B \sqcup C)$. It holds that $A \sqcap B, A \sqcap C \sqsubseteq A$. Then lemma 2.65 implies that $(A \sqcap B) \sqcup (A \sqcap C) \sqsubseteq A$. Also $A \sqcap B \sqsubseteq B$ and $A \sqcap C \sqsubseteq C$. Since $B, C \sqsubseteq B \sqcup C$ by transitivity of $\sqsubseteq$, we have that $A \sqcap B, A \sqcap C \sqsubseteq B \sqcup C$. Then lemma 2.65 implies that $(A \sqcap B) \sqcup (A \sqcap C) \sqsubseteq B \sqcup C$. By applying lemma 2.65 once more, we get that $(A \sqcap B) \sqcup (A \sqcap C) \sqsubseteq A \sqcap (B \sqcup C)$. ☐

Lemma 2.69. Let $A, B \in \mathcal{X}$ be lexicographic sets. Then:

$$A \sqcup B = B \sqcup A \quad \text{and} \quad A \sqcap B = B \sqcap A$$

Proof. Directly implied by lemma 2.53. ☐

Lemma 2.70. Let $A \in \mathcal{X}$ be a lexicographic set. Then:

$$A \sqcup \bot = A \quad \text{and} \quad A \sqcap \top = A$$

Proof. By lemma 2.41, $\bot$ is the minimum element with respect to $\sqsubseteq$ and thus $\bot \sqsubseteq A$. Since we have defined $\sqcup$ as the least upper bound with respect to $\sqsubseteq$ it is implied that $A \sqcup \bot = A$.

Symmetrically, $\top$ is the maximum element with respect to $\sqsubseteq$ and thus $A \sqsubseteq \top$. Since we have defined $\sqcap$ as the greatest lower bound with respect to $\sqsubseteq$ it is implied that $A \sqcap \top = A$. □

Lemma 2.71. Let $B \in \mathcal{X}$ be a lexicographic set. Then if, for all $A \in \mathcal{X}$, $A \sqcup B = A$, then $B = \bot$. Additionally if, for all $A \in \mathcal{X}$, $A \sqcap B = A$, then $B = \top$.

Proof. Let $A \in \mathcal{X}$. Since $A \sqcup B = A$ ($A \sqcap B = A$) and $\sqcup$ ($\sqcap$) is the lub (glb) with respect to $\sqsubseteq$ it is implied that $B \sqsubseteq A$ ($A \sqsubseteq B$). Then it is implied that for all $A \in \mathcal{X}$, $B \sqsubseteq A$ ($A \sqsubseteq B$). Then it holds that $B \sqsubseteq \bot$ ($\top \sqsubseteq B$). By lemma 2.41 it also holds that $\bot \sqsubseteq B$ ($B \sqsubseteq \top$). Since $\sqsubseteq$ is antisymmetric it holds that $B = \bot$ ($B = \top$). □

Lemma 2.72. Let $A, B \in \mathcal{X}$ be lexicographic sets. If $A \sqcup B = \top$ and $A \sqcap B = \bot$, then $B = \overline{A}$.

Proof. First observe that if $A =_0 B$ then lemma 2.53 implies that $A \sqcup B =_0 A \sqcap B$ which of course cannot hold. Then $A \neq_0 B$ must hold. Since $A \sqcup B = \top$, lemma 2.53 implies that $A \mid\mid T_0 \cup B \mid\mid T_0 = X$ and $A \mid\mid F_0 \cap B \mid\mid F_0 = \emptyset$. Also since $A \sqcap B = \bot$, lemma 2.53 implies that $A \mid\mid F_0 \cup B \mid\mid F_0 = X$ and $A \mid\mid T_0 \cap B \mid\mid T_0 = \emptyset$. Also trivially $A \mid\mid T_0 \cap A \mid\mid F_0 = \emptyset$ and $B \mid\mid T_0 \cap B \mid\mid F_0 = \emptyset$. Let $x \in X$. Then $x \in A \mid\mid T_0 \cup B \mid\mid T_0$ and $x \in A \mid\mid F_0 \cup B \mid\mid F_0$. So we have that $x \in A \mid\mid T_0$ iff $x \notin B \mid\mid T_0$ and $x \notin A \mid\mid F_0$ iff $x \in B \mid\mid F_0$. Symmetrically, $x \in A \mid\mid F_0$ iff $x \notin B \mid\mid F_0$ and $x \notin A \mid\mid T_0$ iff $x \in B \mid\mid T_0$. Thus $B \mid\mid T_0 = A \mid\mid F_0$ and $B \mid\mid F_0 = A \mid\mid T_0$. Also $A \mid\mid T_0 \cup A \mid\mid F_0 = X$ and $B \mid\mid T_0 \cup B \mid\mid F_0 = X$. Thus $B = \overline{A}$. □

Remark. Lemma 2.72 cannot be generalized in the same fashion as lemma 2.59. Consider the following counterexample. Let $A = \{\langle a, T_0\rangle, \langle b, F_0\rangle, \langle c, F_1\rangle\}$ and $B = \{\langle a, F_0\rangle, \langle b, T_0\rangle, \langle c, F_1\rangle\}$. Then $A \sqcup B = \{\langle a, T_0\rangle, \langle b, T_0\rangle, \langle c, F_1\rangle\}$ and $A \sqcap B = \{\langle a, F_0\rangle, \langle b, F_0\rangle, \langle c, T_1\rangle\}$. Observe that $A \sqcup B = \overline{A \sqcap B}$ but $B \neq \overline{A}$.

Lemma 2.73. Let $A \in \mathcal{X}$ be a lexicographic set. Then:

$$A \sqcup A = A \quad \text{and} \quad A \sqcap A = A$$

Proof. Directly implied by lemma 2.53. □

Lemma 2.74. Let $A \in \mathcal{X}$ be a lexicographic set. Then:

$$A \sqcup \top = \top \quad \text{and} \quad A \sqcap \bot = \bot$$

Proof. By lemma 2.40, $\top$ is the maximum element with respect to $\sqsubseteq$ and thus $A \sqsubseteq \top$. Since we have defined $\sqcup$ as the least upper bound with respect to $\sqsubseteq$ it is implied that $A \sqcup \top = \top$.

Symmetrically, $\bot$ is the minimum element with respect to $\sqsubseteq$ and thus $\bot \sqsubseteq A$. Since we have defined $\sqcap$ as the greatest lower bound with respect to $\sqsubseteq$ it is implied that $A \sqcap \bot = \bot$. □

Lemma 2.75. Let $A, B \in \mathcal{X}$ be lexicographic sets. Then:

$$A \sqcup (A \sqcap B) = A \quad \text{and} \quad A \sqcap (A \sqcup B) = A$$

Proof. If $A \sqsubseteq B$ then $A \sqcap B = A$ and $A \sqcup A = A$. Suppose that $A \not\sqsubseteq B$. Let μ be the least ordinal such that $A \neq_\mu B$. Then $A \not\sqsubseteq_\mu B$. By definition of $\sqsubseteq_\mu$ either $A \mid\mid T_\mu \not\subseteq B \mid\mid T_\mu$ or $A \mid\mid F_\mu \not\supseteq B \mid\mid F_\mu$. Trivially $A \mid\mid T_\mu \cap B \mid\mid T_\mu \subseteq A \mid\mid T_\mu$ and $A \mid\mid F_\mu \cup B \mid\mid F_\mu \supseteq A \mid\mid F_\mu$.

If $A \mid\mid T_\mu \not\subseteq B \mid\mid T_\mu$ then there exists an $x \in A \mid\mid T_\mu$ such that $x \notin B \mid\mid T_\mu$. Thus $A \mid\mid T_\mu \cap B \mid\mid T_\mu \subset A \mid\mid T_\mu$. Else if $A \mid\mid F_\mu \not\supseteq B \mid\mid F_\mu$ then there exists an $x \in B \mid\mid F_\mu$ such that $x \notin A \mid\mid F_\mu$. Thus $A \mid\mid F_\mu \cup B \mid\mid F_\mu \supset A \mid\mid F_\mu$. In either case by definition of $\sqsubset_\mu$ we have that $A \sqcap B \sqsubset_\mu A$. By lemma 2.53 we have that $A \sqcup (A \sqcap B) = A$.

The proof of the second property is symmetrical. A shorter proof can be obtained using the fact that $\sqcup$ ($\sqcap$) is defined as the least upper bound (greatest lower bound). □

Lemma 2.76. Let $A, B \in \mathcal{X}$ be lexicographic sets. Then:

$$\overline{A \sqcup B} = \overline{A} \sqcap \overline{B} \quad \text{and} \quad \overline{A \sqcap B} = \overline{A} \sqcup \overline{B}$$

Proof. Directly implied by lemmas 2.50 and 2.51. □

Lemma 2.77. Let $A, B \in \mathcal{X}$ be lexicographic sets. The following statements are equivalent:

1. $A \sqsubseteq B$
2. $A \sqcap B = A$
3. $A \sqcup B = B$

Proof. 1 implies 2. Assume that $A \sqsubseteq B$. Then $A \sqcap B = A$ is directly implied since we have defined $\sqcap$ as the greatest lower bound with respect to $\sqsubseteq$.

2 implies 3. Assume that $A \sqcap B = A$. Then $A \sqcup B = (A \sqcap B) \sqcup B = B$ by lemma 2.75.

3 implies 1. Assume that $A \sqcup B = B$. Then since $A \sqsubseteq A \sqcup B$ we have that $A \sqsubseteq B$. □

CHAPTER 3

A MODEL INTERSECTION THEOREM FOR LOGIC PROGRAMS

Before proceeding with the main theorem of this chapter, we first compactly present the necessary definitions from the infinite-valued semantics of [6].

3.1 Infinite-Valued Semantics

Definition 3.1. A normal program rule *is a rule with an atom as head and a conjunction of literals as body*. A normal logic program *is a finite set of normal program rules*.

The infinite-valued treatment follows a common practice which dictates that instead of studying finite first-order logic programs it is more convenient to study their, possibly infinite, *ground instantiations* [2].

Definition 3.2. If P is a normal logic program, its associated *ground instantiation* P^* is constructed as follows: first, put in P^* all ground instances of members of P; second, if a rule $A \leftarrow$ with empty body occurs in P^*, replace it with $A \leftarrow \texttt{true}$; finally, if the ground atom A is not the head of any member of P^*, add $A \leftarrow \texttt{false}$.

Notice that, by construction, P^* is a propositional program. Since the Herbrand Base of normal logic program is countable, P^* has a possibly infinite but countable number of rules. In the rest of this chapter we assume that we study programs that are propositional and have a countable number of rules. Since we are dealing with propositional programs, we often talk about "propositional atoms" and "propositional literals" that appear in the rules of a program.

Definition 3.3. An (infinite-valued) interpretation I of a program P is a function from the set of propositional atoms of P to V.

Observe that in our context, an infinite-valued interpretation, is a lexicographic set in the set of propositional atoms of P.

Definition 3.4. Let I be an interpretation of a program P. Then, I can be extended as follows:

- For every negative atom $\sim p$ appearing in P:

$$I(\sim p) = \begin{cases} T_{\alpha+1}, & \text{if } I(p) = F_\alpha \\ F_{\alpha+1}, & \text{if } I(p) = T_\alpha \\ 0, & \text{if } I(p) = 0 \end{cases}$$

- For every conjunction of literals $l_1, \ldots, l_n$ appearing as the body of a rule in P:

$$I(l_1, \ldots, l_n) = \min\{I(l_1), \ldots, I(l_n)\}$$

Moreover, $I(\texttt{true}) = T_0$ and $I(\texttt{false}) = F_0$.

Finally, the notion of satisfiability of a rule can be defined as follows:

Definition 3.5. Let P be a program and I an interpretation of P. Then, I *satisfies* a rule $p \leftarrow l_1, \ldots, l_n$ of P if $I(p) \geq I(l_1, \ldots, l_n)$. Moreover, I is a *model* of P if I satisfies all rules of P.

3.2 Model Intersection Theorem

For all that follows in this section, let P be a program. Let X be the set of propositional atoms of P. Then $\mathcal{X}$ is the set of all interpretations of P. Let $\mathcal{M} \subseteq \mathcal{X}$ be the set of models of P. First observe that $\mathcal{M} \neq \emptyset$ since $\top$ is trivially a model of P. Also let $\mathcal{N} \subseteq \mathcal{M}$ be an arbitrary non-empty set of models of P. Finally let $(S_\alpha)_{\alpha \leq \kappa}$ be a sequence of lexicographic sets as in definition 2.27 such that $S_\kappa = \bigsqcap \mathcal{N}$.

Lemma 3.6. Let $\alpha < \kappa$ be an ordinal. If $\mathcal{S}_\alpha \neq \emptyset$ then $S_\alpha \in \mathcal{M}$ (i.e. S_α is a model of P).

Proof. Since $\mathcal{S}_\alpha \neq \emptyset$, $S_\alpha = \bigsqcap_\alpha \mathcal{S}_\alpha$. Next we prove that S_α is a model of P. Let $p \leftarrow B$ be a clause of P. We demonstrate that $S_\alpha(p) \geq S_\alpha(B)$. We distinguish cases on the value of $S_\alpha(p)$:

- $S_\alpha(p) = F_\beta$, where $\beta \leq \alpha$. Then there exists a model $M \in \mathcal{S}_\alpha$ such that $M(p) = F_\beta$. Since M is a model of P we have that $M(p) \geq M(B)$. Thus there exists a literal $l \in B$ such that $M(l) \leq F_\beta$. If $l = q$, for some propositional atom q, then $M(q) \leq F_\beta$ and by definition of $\bigsqcap_\alpha$, $M(q) = S_\alpha(q)$ and our claim holds. Otherwise if $l =\sim q$, then it must be that $M(q) > T_\beta$ and again by definition of $\bigsqcap_\alpha$, $M(q) = S_\alpha(q)$ and our claim holds.

- $S_\alpha(p) = T_\beta$, where $\beta \leq \alpha$. Then for any model $M \in \mathcal{S}_\alpha$ we have that $M(p) = T_\beta$. Since M is a model of P we have that $M(p) \geq M(B)$. Thus there exists a literal $l \in B$ such that $M(l) \leq T_\beta$. Now either $l = q$ or $l =\sim q$. If $order(M(q)) < \alpha$ then $M(q) = S_\alpha(q)$ and thus $M(l) = S_\alpha(l)$ and our claim holds. Otherwise, if $l = q$, then $M(q) = T_\alpha$. Then observe that it cannot be that $S_\alpha(q) > T_\alpha$ and for all other possible values of $S_\alpha(q)$ we have that $S_\alpha(q) \leq T_{\alpha+1} < T_\alpha$. If $l =\sim q$, then $F_\alpha \leq M(q) \leq T_\alpha$. Again all other possible values of $S_\alpha(q)$ suffice.

- $S_\alpha(p) = T_{\alpha+1}$. Then by definition of $\bigsqcap_\alpha$, there must exist a model $M \in \mathcal{S}_\alpha$, such that $M(p) < T_\alpha$. Since $M(p) \geq M(B)$ we have that $M(B) < T_\alpha$. Thus a literal $l \in B$ exists such that $M(l) < T_\alpha$. If $l = q$ then, $M(q) < T_\alpha$. Then it cannot be that $S_\alpha(q) \geq T_\alpha$ and any other possible value suffices. If $l =\sim q$ then $M(q) \geq F_\alpha$. Then it cannot be that $S_\alpha(q) < F_\alpha$. Again any other possible value suffices.

Thus in any case $S_\alpha(p) \geq S_\alpha(B)$, so $S_\alpha \in \mathcal{M}$. □

Lemma 3.7. If α is a limit ordinal and for all $\beta < \alpha$, $\mathcal{S}_\beta \neq \emptyset$ then $\bigwedge_{\beta<\alpha} S_\beta \in \mathcal{M}$. (i.e. $\bigwedge_{\beta<\alpha} S_\beta$ is a model of P).

Proof. For all $\beta < \alpha$, since $\mathcal{S}_\beta \neq \emptyset$, we have that $S_\beta = \prod_\beta \mathcal{S}_\beta$. Lemma 3.6 also implies that S_β is a model of P. We prove that $S_\alpha = \bigwedge_{\beta<\alpha} S_\beta$ is a model of P. We know that for any $\beta < \alpha$, $S_\beta =_\beta S_\alpha$ and $S_\alpha \leq S_\beta$. Let $p \leftarrow B$ be a clause of P. We distinguish cases on the value of $S_\alpha(p)$:

- $S_\alpha(p) = F_\beta$, where $\beta < \alpha$. Since $S_\beta =_\beta S_\alpha$ we have that $S_\beta(p) = F_\beta$. Since S_β is a model of P, $S_\beta(p) \geq S_\beta(B)$. Thus a literal $l \in B$ exists, such that $S_\beta(l) \leq F_\beta$. If $l = q$, for some propositional atom q, then $S_\beta(q) \leq F_\beta$. Then since $S_\beta =_\beta S_\alpha$, $S_\alpha(q) \leq F_\beta$ and our claim holds. Otherwise if $l =\sim q$, then it must be that $S_\beta(q) > T_\beta$. Again since $S_\beta =_\beta S_\alpha$, $S_\alpha(q) > T_\beta$ and our claim holds.

- $S_\alpha(p) = T_\beta$, where $\beta < \alpha$. Since $S_\beta =_\beta S_\alpha$ we have that $S_\beta(p) = T_\beta$. Since S_β is a model of P, $S_\beta(p) \geq S_\beta(B)$. Thus a literal $l \in B$ exists, such that $S_\beta(l) \leq T_\beta$. Now either $l = q$ or $l =\sim q$. If $order(S_\beta(q)) \leq \beta$ then since $S_\beta =_\beta S_\alpha$, $S_\alpha(q) = S_\beta(q)$ and our claim holds. Otherwise by definition of $\prod_\beta$, $S_\beta(q) = T_{\beta+1}$. Since $S_\beta =_\beta S_\alpha$ and $S_\alpha \leq S_\beta$ it is implied that $F_{\beta+1} \leq S_\alpha(q) \leq T_{\beta+1}$. Then it holds that $S_\alpha(l) \leq T_{\beta+1}$ and our claim holds.

- $S_\alpha(p) = T_\alpha$ (or $S_\alpha(p) = 0$, if $\alpha = \kappa$). Then for all $\beta < \alpha$, $S_\beta(p) = T_{\beta+1}$. Consider any $\beta < \alpha$. It holds that $S_\beta(p) \geq S_\beta(B)$. Thus there exists a literal $l \in B$ such that $S_\beta(l) \leq T_{\beta+1}$. Also $l = q$ or $l =\sim q$ for some $q \in B_P$. By definition of $\prod_\beta$, if $l = q$ then, either $S_\beta(q) \leq F_\beta$ or $S_\beta(q) = T_{\beta+1}$, otherwise if $l =\sim q$ then, either $S_\beta(q) = F_\beta$ or $S_\beta(q) = T_{\beta+1}$ or $S_\beta(q) \geq T_\beta$. If there exists a $\beta < \alpha$ such that $l = q$ and $S_\beta(q) \leq F_\beta$ or $l =\sim q$ and $S_\beta(q) \geq T_\beta$ our claim holds since $S_\beta =_\beta S_\alpha$.

  Otherwise suppose that for all ordinals $\beta < \alpha$, it holds that $S_\beta(q) = T_{\beta+1}$. Then observe that if $S_\beta(q) = T_{\beta+1}$ then for all $\gamma < \beta$, by definition of $\prod_\gamma$, it must hold that $S_\gamma(q) = T_{\gamma+1}$. This implies that there must exist a common literal $l \in B$, for whose corresponding propositional atom q, it holds that $S_\beta(q) = T_{\beta+1}$, for all $\beta < \kappa$. This in turn implies that $S_\alpha(q) = glb\{S_\beta(q) \mid \beta < \alpha\} = glb\{T_{\beta+1} \mid \beta < \alpha\} = T_\alpha$ (or 0). Thus our claim holds.

  Finally suppose that it is not the case that for all ordinals $\beta < \alpha$, $S_\beta(q) = T_{\beta+1}$ holds. Then there exists some ordinal $\beta < \alpha$ for which there exists a literal $l =\sim q$ such that $S_\beta(q) = F_\beta$ and $S_\beta(l) < S_\beta(l')$, for any other literal $l' \in B$. Now observe that since $S_\beta(q) = F_\beta$ then for any ordinal γ, such that $\beta < \gamma$, since $S_\beta =_\beta S_\gamma$, it holds that $S_\gamma(q) = F_\beta$ and thus $S_\gamma(p) < S_\gamma(l)$. This implies that there must exist a different literal l' such that $S_\gamma(p) \geq S_\gamma(l')$. Since each time this case occurs a literal in B is ``used up" and B is finite by definition while between any $\beta < \alpha$ there are at least countably infinite many ordinals, there must exist an ordinal δ, where $\beta < \delta < \alpha$ such that there exists a literal l''

and its corresponding atom r, such that $S_\delta(r) = T_{\delta+1}$. As previously stated this implies that $S_\beta(r) = T_{\beta+1}$. Observe that then $S_\beta(l'') \leq S_\beta(l)$ which contradicts our assumption.

In any case $S_\alpha(p) \geq S_\alpha(B)$ and thus $\bigwedge_{\beta<\alpha} S_\beta$ is a model of P. □

Theorem 3.8. Let P be a program. Let $\mathcal{N}$ be an arbitrary non-empty set of models of P. Then $\bigsqcap \mathcal{N}$ is a model of P.

Proof. Let $\delta < \kappa$ be the least ordinal such that $\mathcal{S}_\delta = \emptyset$. If such an ordinal does not exist then let $\delta = \kappa$. Then for any ordinal ϵ, such that $\delta < \epsilon < \kappa$, by definition of $\mathcal{S}_\epsilon$, it holds that $\mathcal{S}_\epsilon = \emptyset$. Then $S_\kappa = \bigwedge_{\alpha<\kappa} S_\alpha = S_\delta$, since we have previously proved that for all $\alpha < \delta$, $S_\delta \leq S_\alpha$.

If δ is a successor ordinal then it is of the form $\delta = \delta' + 1$. Then $\mathcal{S}_{\delta'} \neq \emptyset$ and by lemma 3.6, $S_{\delta'}$ is a model of P. Then $S_\delta = \bigwedge_{\alpha<\delta} S_\alpha = S_{\delta'}$. Thus S_κ is a model of P.

If δ is a limit ordinal then by the hypothesis, for all $\alpha < \delta$ we have that $\mathcal{S}_\alpha \neq \emptyset$. Then by lemma 3.7, $S_\delta = \bigwedge_{\alpha<\delta} S_\alpha$ is a model of P. Thus S_κ is a model of P.

In any case $\bigsqcap \mathcal{N} = S_\kappa$ is a model of P. □

Corollary 3.9. Let P be a program. Let $\mathcal{M}$ be the set of models of P. Then $\bigsqcap \mathcal{M}$ is the *minimum model* of P with respect to the lexicographic subset relation $\sqsubseteq$.

Proof. We have defined $\bigsqcap$ as the greatest lower bound with respect to $\sqsubseteq$. Then $\bigsqcap \mathcal{M} \sqsubseteq \mathcal{M}$. It remains to show that $\bigsqcap \mathcal{M} \in \mathcal{M}$. But that is directly implied by Theorem 3.8, when $\mathcal{N} = \mathcal{M}$. □

BIBLIOGRAPHY

[1] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2 edition, 2002.

[2] Melvin Fitting. Fixpoint semantics for logic programming a survey. *Theoretical Computer Science*, 278(1):25 -- 51, 2002. Mathematical Foundations of Programming Semantics 1996.

[3] J.A Goguen. L-fuzzy sets. *Journal of Mathematical Analysis and Applications*, 18(1):145 -- 174, 1967.

[4] E. E. Kerre. A first view on the alternatives of fuzzy set theory. In Bernd Reusch and Karl-Heinz Temme, editors, *Computational Intelligence in Theory and Practice*, pages 55--71, Heidelberg, 2001. Physica-Verlag HD.

[5] J. W. Lloyd. Foundations of logic programming. *Journal of Symbolic Logic*, 52(1):288--289, 1987.

[6] Panos Rondogiannis and William W. Wadge. Minimum model semantics for logic programs with negation-as-failure. *ACM Trans. Comput. Logic*, 6(2):441–467, April 2005.

[7] M. H. Van Emden and R. A. Kowalski. The semantics of predicate logic as a programming language. *J. ACM*, 23(4):733–742, October 1976.

[8] L.A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338 -- 353, 1965.

[9] Zoltán Ésik and Panos Rondogiannis. A fixed point theorem for non-monotonic functions. *Theoretical Computer Science*, 574:18 -- 38, 2015.