Non-constructive proof of a fixed point theorem on lexicographic lattice structures

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ABSTRACT

In this thesis, we develop a novel, non-constructive proof of the fixed point theorem proposed by A. Charalambidis, G. Chatziagapis and P. Rondogiannis [1]. This theorem proves the existence of a least fixed point of functions that possess a restricted form of monotonicity, and are defined over some specially structured partially ordered sets, which we will call *lexicographic lattice structures*. Our results give as a special case the well-known Knaster-Tarski theorem when restricted to monotonic functions.

The initial proof of the theorem, as presented at LICS 2020, is constructive. Our novel proof is simpler and it gives an alternative intuition and a deeper understanding of the theorem. Furthermore, we prove that the sets of pre-fixed, post-fixed and fixed points of function over those structures form a complete lattice. Our proofs have been verified through the Coq proof assistant. Our results have direct applications in fields of Computer Science where non-monotonic formalisms are being used, such as Artificial Intelligence, Logic Programming and Formal Language Theory.

ΣΥΝΟΨΗ

Στη διπλωματική αυτή αναπτύσσουμε μια νέα, μη-κατασκευαστική απόδειξη του θεωρήματος σταθερού σημείου, που προτάθηκε από τους Α. Χαραλαμπίδη, Γ. Χατζηαγάπη, και Π. Ροντογιάννη [1]. Το θεώρημα αυτό αφορά στην ύπαρξη ελάχιστου σταθερού σημείου μιας κλάσης συναρτήσεων που διαθέτουν ένα περιορισμένο είδος μονοτονικότητας, και οι οποίες είναι ορισμένες σε ειδικώς δομημένα μερικώς διατεταγμένα σύνολα, τα οποία ονομάζουμε *δομές λεξικογραφικού πλέγματος*. Όταν το θεώρημα εφαρμόζεται σε μονοτονικές συναρτήσεις, δίνει ως ειδική περίπτωση το κλασικό θεώρημα των Knaster-Tarski.

Η αρχική απόδειξη του θεωρήματος, όπως παρουσιάζεται στο LICS 2020, είναι κατασκευαστική. Η προτεινόμενη απόδειξη είναι πιο απλή από την αρχική και δίνει μια εναλλακτική διαίσθηση και περαιτέρω εμβάθυνση στο θεώρημα σταθερού σημείου. Επιπροσθέτως, αποδεικνύουμε ότι τα σύνολα των προ-σταθερών, μετα-σταθερών, και σταθερών σημείων των συναρτήσεων πάνω σε αυτές τις δομές, σχηματίζουν πλήρη πλέγματα. Οι αποδείξεις μας έχουν επαληθευτεί μέσω του Coq. Τα αποτελέσματά μας έχουν άμεσες εφαρμογές σε περιοχές της Πληροφορικής, όπου χρησιμοποιούνται μη-μονοτονικοί φορμαλισμοί, όπως στην Τεχνητή Νοημοσύνη, στο Λογικό Προγραμματισμό και στη Θεωρία Τυπικών Γλωσσών.

CONTENTS

CHAPTER 1 INTRODUCTION

Negation-as-failure is one of the many attempts that have been made in order to extend logic programming. Intuitively negation-as-failure means that in order to determine if $not\ p$ is valid, one has to examine whether p can be proved. If the process of proving p terminates and fails, $not\ p$ is valid. Otherwise, if p holds, then $not\ p$ fails. Even though negation-as-failure is trivial to implement, it is extremely difficult to formalize from a semantic point of view. The primary reason for this is that negation is in general non-monotonic.

A huge leap in capturing the meanings of negation-as-failure was the introduction of the *well-founded semantics* [7], which utilizes a three-valued logic in order to formalize the meaning of a logic program with negation. After that, P. Rondogiannis. and W. Wadge presented a new approach [11] based on an infinite-valued logic. This approach is important because it leads to a unique minimum model in a program-independent ordering. Moreover, if we restrict this model to a three-valued logic we get the well-founded model.

It is shown in [11] that the minimum model is the *least fixed point* of a non-monotonic operator with respect to an ordering relation. In [6] only the set-theoretic essence is kept, the logic programming related issues are omitted and some sufficient conditions are presented for a fixed point to exist.

In [6] two ordering relations are being used over a set L, namely $\leq$ and $\sqsubseteq$. The first one corresponds to a "pointwise" comparison, whereas the second to a "lexicographic" one. $(L, \leq)$ is supposed to be a complete a lattice and $(L, \sqsubseteq)$ is later shown to form a complete lattice. In my bachelor thesis [3] and in [1], we used only the "lexicographic" comparison and we supposed that it forms a complete lattice. Omitting the first ordering leads to a simpler proof and it becomes more clear that this theorem is a generalization of the Knaster-Tarski theorem.

The rest of the paper is organized as follows: Chapter 2 defines the complete prelattices, a generalization of complete lattices. Chapter 3 defines the specially structured complete lattices that will be the objects of our study and investigates some properties of these lattices. Chapter 4 presents and proves the fixed point theorem. Chapter 5 establishes some properties of the pre-fixed, post-fixed and fixed points of functions over lexicographic lattice structures. Chapter 6 presents two applications of the theoretical results obtained in the paper. Finally, Chapter 7 concludes the study.

In the following, we assume familiarity with the basic notions of partially ordered sets and particularly lattices (such as for example [5]).

CHAPTER 2

COMPLETE PRELATTICES

In mathematics, a preorder is a binary relation that is reflexive and transitive. In this chapter, we will define as a *complete prelattice* a preordered set in which all subsets have both a least upper bound and a greatest lower bound. The complete prelattices are very similar to complete lattices, the only difference is that the binary relation of a complete prelattice is not necessarily antisymmetric.

Definition 2.1. Suppose $\lesssim$ is a binary relation over a set S. $(S, \lesssim)$ is called a *preorder* iff $\lesssim$ is reflexive and transitive.

Definition 2.2. Suppose S is a set and $\sim$ is a binary relation over that set. For all $x \in S$ and for each $Y, Z \subseteq S$ we will write:

- $x \sim Y$ iff $x \sim y$ for all $y \in Y$.
- $Y \sim x$ iff $y \sim x$ for all $y \in Y$.
- $Y \sim Z$ iff $y \sim z$ for all $y \in Y$ and $z \in Z$.

Definition 2.3. Let $(S, \lesssim)$ be a preorder, $X \subseteq S$ and $x \in S$. x is called:

- a $\lesssim$-*bottom element* of S iff $x \lesssim S$.
- a $\lesssim$-*top element* of S iff $S \lesssim x$.
- an $\lesssim$-*upper bound* of X iff $X \lesssim x$.
- a $\lesssim$-*lower bound* of X iff $x \lesssim X$.
- a $\lesssim$-*least upper bound* of X iff x is an $\lesssim$-upper bound of X and for each $y \in S$ such that y is an $\lesssim$-upper bound of X, $x \lesssim y$.
- a $\lesssim$-*greatest lower bound* of X iff x is a $\lesssim$-lower bound of X and for each $y \in S$ such that y is a $\lesssim$-lower bound of X, $y \lesssim x$.

Definition 2.4. Let $(S, \lesssim)$ be a preorder. $(S, \lesssim)$ is called a *complete prelattice* iff for any $X \subseteq S$, X has a $\lesssim$-least upper bound and a $\lesssim$-greatest lower bound in S.

Lemma 2.5. Let $(S, \lesssim)$ be a preorder, such that S has a $\lesssim$-bottom element and for every non-empty $X \subseteq S$, X has a $\lesssim$-least upper bound in S. Then $(S, \lesssim)$ is a complete prelattice.

Proof. Suppose $X \subseteq S$. If $X \neq \emptyset$, by the hypothesis S has a $\lesssim$-least upper bound in S. If $X = \emptyset$, every element of S is an $\lesssim$-upper bound of X, so that the $\lesssim$-bottom element of S is a $\lesssim$-least upper bound of X. Suppose now, $LB = \{s \in S : s \lesssim X\}$. LB contains the $\lesssim$-bottom element of S, so that $LB \neq \emptyset$, and thus, by the hypothesis, has a $\lesssim$-least upper bound in S, say y. Suppose $x \in X$. We have that $LB \lesssim x$, so that x is an $\lesssim$-upper bound of LB, and thus $y \lesssim x$. Since that holds for any $x \in X$, y is a $\lesssim$-lower bound of X. Let z be a $\lesssim$-lower bound of X. We have that $z \in LB$, and therefore, $z \lesssim y$. Thus, y is a $\lesssim$-greatest lower bound of X. □

Lemma 2.6. Let $(S, \lesssim)$ be a complete prelattice and $v \in S$. Then $(\{x \in S : v \lesssim x\}, \lesssim)$ is a complete prelattice.

Proof. Let $S' = \{x \in S : u \lesssim v\}$. It is easy to show that any subset of a preorder is a preorder. In order to prove that $(S', \lesssim)$ is a complete prelattice, by Lemma 2.5, it is sufficient to prove that S' has a $\lesssim$-bottom element and that every non-empty subset $X \subseteq S'$ has a $\lesssim$-least upper bound in S'. Obviously v is a $\lesssim$-bottom element of S'. Suppose non-empty $X \subseteq S'$ and $x \in X$. Since $X \subseteq S$, X has a $\lesssim$-least upper bound in S, say y. We have that $v \lesssim x \lesssim y$. By the transitivity of $\lesssim$, $v \lesssim x$, so that $y \in S'$. □

Definition 2.7. Let S be a set and $\sim \subseteq S^2$. A function $f : S \to S$ is called *$\sim$-monotonic* iff for all $x, y \in S$, $x \sim y$ implies $f(x) \sim f(y)$.

Theorem 2.8. Let $(S, \lesssim)$ be a complete prelattice and $f : S \to S$ be a $\lesssim$-monotonic function over S. Then, there exists some $x \in S$ such that $x \lesssim f(x)$, $f(x) \lesssim x$ and for every $y \in S$ such that $f(y) \lesssim y$, $x \lesssim y$.

Proof. Let $P = \{s \in S : f(s) \lesssim s\}$ and x be a $\lesssim$-greatest lower bound of P. Suppose $z \in P$. We have that $f(z) \lesssim z$. Since x is a $\lesssim$-lower bound of P, $x \lesssim z$. Since f is $\lesssim$-monotonic, we have that $f(x) \lesssim f(z)$, and by the transitivity of $\lesssim$, $f(x) \lesssim z$. Thus, $f(x)$ is a $\lesssim$-lower bound of P, so that $f(x) \lesssim x$. By the $\lesssim$-monotonicity of f, $f(f(x)) \lesssim f(x)$, so that $f(x) \in P$, and thus $x \lesssim f(x)$. Suppose now $y \in S$ such that $f(y) \lesssim y$. We have that $y \in P$, so that $x \lesssim y$. □

CHAPTER 3

LEXICOGRAPHIC LATTICE STRUCTURES

In this chapter we will define the lexicographic lattice structures, which are the objects of our study. Then we will investigate some of their properties.

3.1 Definitions

Consider a complete lattice $(L, \sqsubseteq)$, whose least element will be denoted by $\perp$ and the lub and glb operations by $\bigsqcup$ and $\bigsqcap$ respectively. In order to define the notion of a *lexicographic lattice structure*, we assume that $\sqsubseteq$ can be "constructed" using a sequence of preorderings $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$, where $\kappa > 0$ is an ordinal. Actually, as we are going to see, *every* complete lattice $(L, \sqsubseteq)$ can be "constructed" in a trivial way using such preorderings; using this trivial construction, we will be able to obtain as a special case of our theorem the well-known Knaster-Tarski fixed point theorem. Of course, we will be mostly interested in the case where $\sqsubseteq$ is "constructed" in a non-trivial way from the preorderings; in this case our fixed point theorem will be applicable to a much broader class of functions, namely functions that are potentially non-monotonic.

Before giving any formal definitions, we present the intuition behind the above notions, using a well-known example. Let us take L to be the set of ω-words (ie., $\kappa = \omega$ in our example) over a finite alphabet Σ. We assume that the elements of Σ are alphabetically ordered. Let us take the $\sqsubseteq$ relation to be the lexicographic comparison of ω-words. One can easily verify that the set of ω-words under the lexicographic ordering, is a complete lattice. Consider now for each $\alpha < \omega$, the preordering $\sqsubseteq_\alpha$ to be the relation that compares two ω-words up to their α-th elements: given two ω-words x and y, we write $x \sqsubset_\alpha y$ iff x and y are identical at all positions less than α and the sequence x contains at position α a character of Σ that is alphabetically "smaller" than the corresponding character of y at the same position. Notice now that the lexicographic ordering $\sqsubseteq$ can be constructed using the relations $\sqsubseteq_\alpha$: given two ω-words x and y, it holds $x \sqsubset y$ iff there exists some α such that $x \sqsubset_\alpha y$. In the following, we formalize the above ideas. We start with some simple notational conventions:

Definition 3.1. Let $(L, \sqsubseteq)$ be a complete lattice. Let $\kappa > 0$ be an ordinal and let $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ be a sequence of preorderings over L. For all $x, y \in L$, we write $x \sqsubset y$ if $x \sqsubseteq y$ and $x \neq y$. For every $\alpha < \kappa$ and $x, y \in L$, we write $x =_\alpha y$ iff $x \sqsubseteq_\alpha y$ and $y \sqsubseteq_\alpha x$.

We write $x \sqsubset_\alpha y$ iff $x \sqsubseteq_\alpha y$ but $x =_\alpha y$ does not hold. We write $x \approx_\alpha y$ iff $x =_\beta y$ for all $\beta < \alpha$. For all $\alpha < \kappa$, we define $(x]_\alpha = \{y \in L : x \approx_\alpha y\}$; moreover, for all $\alpha < \kappa$ we define $[x]_\alpha = \{y \in L : x =_\alpha y\}$.

In our setting, we will insist that the partial order $\sqsubseteq$ and the preorderings $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ are closely related in the sense that the latter relations determine the former one:

Definition 3.2. Let $(L, \sqsubseteq)$ be a complete lattice. Let $\kappa > 0$ be an ordinal and let $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ be a set of preorderings over L. We will say that the relation $\sqsubseteq$ is *determined* by the preorderings $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ if for all $x, y \in L$, $x \sqsubset y$ iff $x \sqsubset_\alpha y$ for some $\alpha < \kappa$.

We can now define lexicographic lattice structures:

Definition 3.3. Let $(L, \sqsubseteq)$ be a complete lattice. Let $\kappa > 0$ be an ordinal, let $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$ be a set of preorderings over L, and assume that $\sqsubseteq$ is determined by these preorderings. The triple $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa}\rangle$ will be called a *lexicographic lattice structure* if the following three properties hold:

Property 1. For every $\alpha < \kappa$ and for all $x, y \in L$, if $x \sqsubseteq_\alpha y$, then $x \approx_\alpha y$.

Property 2. For all $x, y \in L$, if $x \approx_\kappa y$, then $x = y$.

Property 3. For every $x \in L$ and for every ordinal $\alpha < \kappa$, $\bigsqcup [x]_\alpha =_\alpha x$ and $\bigsqcap [x]_\alpha =_\alpha x$.

The intuition behind the above definition can be outlined as follows. First of all, one can think of the elements of L as entities consisting of κ levels. More generally, Property 1 states that each successive preordering relation provides a more accurate comparison of the elements of L. Property 2 states that if two elements of L are indistinguishable with respect to all our preordering relations, then the two elements must coincide. Finally, Property 3 states that if we consider the set of elements of L that have the same "prefix" until their stratum α, then this set has a least and a greatest element.

One can verify that there exist several natural application domains in which lexicographic lattice structures can be used. One of the most natural ones, is the set of ω-words discussed earlier in this section. We can now state this in a more formal way. Given a finite alphabet Σ whose elements are ordered by a relation $<$, consider the triple $\langle \Sigma^\omega, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\omega}\rangle$, where, for all $x, y \in \Sigma^\omega$ and for all $\alpha < \omega$:

$$x \sqsubset_\alpha y \text{ iff } [\forall \beta < \alpha (x(\beta) = y(\beta)) \land (x(\alpha) < y(\alpha))]$$

and

$$x \sqsubset y \text{ iff } \exists \alpha < \omega [\forall \beta < \alpha (x(\beta) = y(\beta)) \land (x(\alpha) < y(\alpha))]$$

It is not hard to check that the requirements of Definition 3.3 are all satisfied. In particular, Property 3 holds because for every ω-sequence x and every $\alpha < \omega$, the $\sqsubseteq$-least (respectively, $\sqsubseteq$-greatest) element of $[x]_\alpha$ is the sequence that is identical to x at all indices $\beta \leq \alpha$ and at all indices that are greater than α it has a constant value that coincides with the alphabetically least (respectively, greatest) element of Σ.

More generally, the application domains in which lexicographic lattice structures appear to be applicable, are sets that have a natural stratification and are accompanied by a natural lexicographic ordering. Indicatively, we mention the set of infinite-valued interpretations of logic programs with negation [11], the set of interpretations of higher-order logic programs with negation [2], the set of interpretations of boolean grammars [10, 8], the set of transfinite sequences over complete lattices [4], and so on. Two of these applications will be presented in detail in Chapter 6.

Remark 1. For *every* complete lattice $(L, \sqsubseteq)$ we can create a trivial lexicographic lattice structure: simply take $\kappa = 1$ and $\sqsubseteq_0$ to be equal to $\sqsubseteq$. Obviously, $\sqsubseteq_0$ determines $\sqsubseteq$ (because they coincide). Moreover, the triple $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<1}\rangle = \langle L, \sqsubseteq, \{\sqsubseteq\}\rangle$ satisfies the three properties of Definition 3.3.

Remark 2. In our proofs we can safely assume that the ordinal κ in the definition of lexicographic lattice structures, is always a *limit* one. The formal justification for this remark is identical to the one given in [6][page 23]. Intuitively, given a lexicographic lattice structure $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa}\rangle$, where κ is a successor ordinal, we can create a structure $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\lambda}\rangle$, where λ is the least limit ordinal that is greater than κ, and the relations $\sqsubseteq_\beta$, for $\kappa \leq \beta < \lambda$, are all equal to the identity relation over L. It can be easily seen that the new structure is a lexicographic lattice one (ie., it satisfies the properties of Definition 3.3) and it can be used interchangeably with the initial structure for the purposes of the paper.

Remark 3. Let $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa}\rangle$ be a lexicographic lattice structure. For all $x, y \in L$ and $\alpha < \kappa$, let us define $x \sqsupseteq y$ iff $y \sqsubseteq x$ and $x \sqsupseteq_\alpha y$ iff $y \sqsubseteq_\alpha x$. The *dual of a proposition* is obtained by replacing each occurrence of $\sqsubseteq$ by $\sqsupseteq$, each occurrence of $\sqsubseteq_\alpha$ by $\sqsupseteq_\alpha$, each occurrence of $\bigsqcup$ by $\bigsqcap$, each occurrence of $\bigsqcup_\alpha$ by $\bigsqcap_\alpha$, and finally, for any relation $\circ$ each occurrence of $\circ$-greatest by $\circ$-least and each occurrence of $\circ$-least by $\circ$-greatest. It is clear that all of the properties in Definition 3.3 imply the dual of themselves. For each proposition P that can be proved using these properties, the dual of P can be established by the dual of the proof of P. Therefore, our theory is closed under dual operation.

In the rest of the paper, we will assume that we have a fixed lexicographic lattice structure $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa}\rangle$. This will allow us to use κ, L, $\sqsubseteq$, and the preorderings $\sqsubseteq_\alpha$ freely in lemmas, definitions, and so on (avoiding in this way to repeat statements such as "Let $\kappa > 0$ be an ordinal, let L be a complete lattice, ...", and so on).

3.2 Some Consequences of the Properties

Lemma 3.4. Let $\alpha < \kappa$ and let $x, y, z \in L$ such that $x \sqsubset_\alpha y$ and $y \sqsubseteq_\alpha z$. Then, $x \sqsubset_\alpha z$.

Proof. Since $x \sqsubset_\alpha y$, we also have $x \sqsubseteq_\alpha y$. By the transitivity of $\sqsubseteq_\alpha$, $x \sqsubseteq_\alpha z$. Suppose, for the sake of contradiction, that $z \sqsubseteq_\alpha x$. By the transitivity of $\sqsubseteq_\alpha$, $y \sqsubseteq_\alpha x$ (contradiction). Thus, $x \sqsubset_\alpha z$. □

Lemma 3.5. Let $\alpha < \kappa$ and suppose $x, y \in L$. If $x \sqsubseteq y$ and $x \approx_\alpha y$ then $x \sqsubseteq_\alpha y$.

Proof. By the definition of $\sqsubseteq$, we have $x = y$ or $x \sqsubset_\beta y$ for some $\beta < \kappa$. If $x = y$, by Property 2 we get $x =_\alpha y$ which implies that $x \sqsubseteq_\alpha y$. If $x \sqsubset_\beta y$ for some β, then it must be the case that $\beta \geq \alpha$, because $x =_\beta y$ for all $\beta < \alpha$. If $\beta = \alpha$, then $x \sqsubset_\alpha y$, so $x \sqsubseteq_\alpha y$ clearly holds. If $\beta > \alpha$, by Property 1 $x =_\alpha y$, so that $x \sqsubseteq_\alpha y$ again. □

Lemma 3.6. Let $\alpha < \kappa$, non-empty $X \subseteq L$ and $x \in L$. If $X \approx_\alpha x$ then $\bigsqcup X \approx_\alpha x$ and $\bigsqcap X \approx_\alpha x$.

Proof. We are going to prove that $\bigsqcup X \approx_\alpha x$. By duality, we can prove similarly that $\bigsqcap X \approx_\alpha x$. Let z be an element in X. It is sufficient to prove that $\bigsqcup X =_\beta z$ for all $\beta < \alpha$. We have that $z \sqsubseteq \bigsqcup X$. If $z = \bigsqcup X$ or $z \sqsubset_\beta \bigsqcup X$ for some $\beta \geq \alpha$ we are done.

Assume, for the sake of contradiction, that $z \sqsubset_\beta \bigsqcup X$ for some $\beta < \alpha$. Let $y = \bigsqcup [z]_\beta$. Clearly $X \subseteq [z]_\beta$, so that y is a $\sqsubseteq$-upper bound of X and therefore, $\bigsqcup X \sqsubseteq y$. By Property 3, $y =_\beta z$. By Lemma 3.4, $y \sqsubset_\beta \bigsqcup X$, so that $y \sqsubset \bigsqcup X$, which contradicts $\bigsqcup X \sqsubseteq y$. ☐

Definition 3.7. Let $\alpha \leq \kappa$. A set $X \subseteq L$ is called $\approx_\alpha$*-closed* iff for all $x, y \in X$, $x \approx_\alpha y$.

Lemma 3.8. Let $\alpha < \kappa$ and $X \subseteq L$, such that X is non-empty and $\approx_\alpha$-closed. Then $\bigsqcap X \sqsubseteq_\alpha X \sqsubseteq_\alpha \bigsqcup X$.

Proof. Suppose $x \in X$. Since X is $\approx_\alpha$-closed, $X \approx_\alpha x$. By Lemma 3.6, $\bigsqcap X \approx_\alpha x \approx_\alpha \bigsqcup X$. Since $\bigsqcap X \sqsubseteq x \sqsubseteq \bigsqcup X$, by Lemma 3.5, $\bigsqcap X \sqsubseteq_\alpha x \sqsubseteq_\alpha \bigsqcup X$ ☐

Lemma 3.9. Let $\alpha < \kappa$ and $X \subseteq L$, such that X is non-empty and $\approx_\alpha$-closed. For any $x \in L$ such that $X \sqsubseteq_\alpha x$, $\bigsqcup X \sqsubseteq_\alpha x$.

Proof. Suppose non-empty and $\approx_\alpha$-closed $X \subseteq L$ and $x \in L$ such that $X \sqsubseteq_\alpha x$. Let $X_1 = X \cap [x]_\alpha$, $X_2 = X \setminus X_1$ and y be the $\sqsubseteq$-greatest element of $[x]_\alpha$. Clearly $X_2 \sqsubset_\alpha x$. We have that $X_1 \sqsubseteq y$ and $X_2 \sqsubset x \sqsubseteq y$, so that y is a $\sqsubseteq$-upper bound of X. Thus $\bigsqcup X \sqsubseteq y$. Since $X \sqsubseteq_\alpha x$, by Property 1 $X \approx_\alpha x$. By Lemma 3.6, $\bigsqcup X \approx_\alpha z$. Also by Property 3, $x =_\alpha y$, so that, by Property 1, $x \approx_\alpha y$. Therefore, $\bigsqcup X \approx_\alpha y$. By Lemma 3.5 $\bigsqcup X \sqsubseteq_\alpha y$ and by the transitivity of $\sqsubseteq_\alpha$, $\bigsqcup X \sqsubseteq_\alpha x$. ☐

Lemma 3.10. Let $\alpha \leq \kappa$ be a limit ordinal and $\{X_\beta\}_{\beta<\alpha}$ be a sequence of subsets of L, such that for all $\beta < \alpha$, X_β is $\approx_\beta$-closed. Then $\bigcap_{\beta<\alpha} X_\beta$ is $\approx_\alpha$-closed.

Proof. If $\bigcap_{\beta<\alpha} X_\beta = \emptyset$ has less than two elements, it is trivially $\approx_\alpha$-closed. Otherwise, suppose $x, y \in \bigcap_{\beta<\alpha} X_\beta$. For every $\beta < \alpha$, $x, y \in X_{\beta+1}$, and because $X_{\beta+1}$ is $\approx_{\beta+1}$-closed, $x =_\beta y$, so we have that $\bigcap_{\beta<\alpha} X_\beta$ is $\approx_\alpha$-closed. ☐

Definition 3.11. A non-empty set $X \subseteq L$ is called *complete* iff for every non-empty $Y \subseteq X$, $\bigsqcup Y \in X$ and $\bigsqcap Y \in X$.

Lemma 3.12. Let $\alpha < \kappa$, $X \subseteq L$ and $x \in X$. If X is complete, then $X \cap [x]_\alpha$ is complete.

Proof. $X \cap [x]_\alpha$ is non-empty, since $x \in X \cap [x]_\alpha$. Suppose non-empty $Y \subseteq X \cap [x]_\alpha$. We have that $Y \subseteq X$ and $Y \subseteq [x]_\alpha$. Since X is complete, $\bigsqcup Y \in X$ and $\bigsqcap Y \in X$. Also, $Y \approx_\alpha x$, so that, by Lemma 3.6, $\bigsqcup Y \approx_{\alpha+1} x$ and $\bigsqcap Y \approx_{\alpha+1} x$, and thus, $\bigsqcup Y \in [x]_\alpha$ and $\bigsqcap Y \in [x]_\alpha$. Therefore, $\bigsqcup Y \in X \cap [x]_\alpha$ and $\bigsqcap Y \in X \cap [x]_\alpha$. ☐

Lemma 3.13. Let $\alpha < \kappa$ and $X \subseteq L$. If X is complete and $\approx_\alpha$-closed, then $(X, \sqsubseteq_\alpha)$ is a complete prelattice.

Proof. $(X, \sqsubseteq_\alpha)$ is a preorder, since $\sqsubseteq_\alpha$ is reflexive and transitive over L. In order to prove that $(X, \sqsubseteq_\alpha)$ is a complete prelattice, by Lemma 2.5, it is sufficient to prove that X has a $\sqsubseteq_\alpha$-bottom element and that every non-empty subset $Y \subseteq X$ has a $\sqsubseteq_\alpha$-least upper bound in X. Since X is complete, $\bigsqcap X \in X$, and by Lemma 3.8 $\bigsqcap X$ is a $\sqsubseteq_\alpha$-bottom element of X. Suppose non-empty $Y \subseteq X$. Since X is complete, $\bigsqcup Y \in X$. By Lemma 3.8 and Lemma 3.9, $\bigsqcup Y$ is a $\sqsubseteq_\alpha$-least upper bound of Y. ☐

Lemma 3.14. Let $\alpha \leq \kappa$ be a limit ordinal and $\{X_\beta\}_{\beta<\alpha}$ be a sequence of complete subsets of L, such that for all $\beta < \gamma < \alpha$, and $X_\beta \supseteq X_\gamma$. Then $\bigcap_{\beta<\alpha} X_\beta \neq \emptyset$.

Proof. For every $\beta < \alpha$, let $x_\beta = \bigsqcap X_\beta$. Let $x = \bigsqcup\{x_\beta : \beta < \alpha\}$. Whenever $\beta < \gamma < \alpha$, since X_γ is complete, we have that $x_\gamma \in X_\gamma$. Moreover, $X_\gamma \subseteq X_\beta$, so that $x_\gamma \in X_\beta$. Thus $x_\beta \sqsubseteq x_\gamma$ whenever $\beta < \gamma < \alpha$. Therefore, for each $\beta < \alpha$, $x = \bigsqcup\{x_\gamma : \beta < \gamma < \alpha\}$. Let $\beta < \alpha$. Since for any γ such that $\beta < \gamma < \alpha$, $x_\gamma \in X_\beta$, we have that $\{x_\gamma : \beta < \gamma < \alpha\} \subseteq X_\beta$. Since X_β is complete and $\{x_\gamma : \beta < \gamma < \alpha\}$ is non-empty, because α is a limit ordinal, $x \in X_\beta$. Since this holds for any $\beta < \alpha$, $x \in \bigcap_{\beta<\alpha} X_\beta$. □

Lemma 3.15. Let $\alpha \leq \kappa$ be a limit ordinal and $\{X_\beta\}_{\beta<\alpha}$ be a sequence of complete subsets of L. If $\bigcap_{\beta<\alpha} X_\beta \neq \emptyset$ then $\bigcap_{\beta<\alpha} X_\beta$ is complete.

Proof. Suppose non-empty $Y \subseteq \bigcap_{\beta<\alpha} X_\beta$. For any $\beta < \alpha$ we have that $Y \subseteq X_\beta$, and because X_β is complete, $\bigsqcup Y \in X_\beta$ and $\bigsqcap Y \in X_\beta$. Since this holds for any $\beta < \alpha$, $\bigsqcup Y \in \bigcap_{\beta<\alpha} X_\beta$ and $\bigsqcap Y \in \bigcap_{\beta<\alpha} X_\beta$. □

Definition 3.16. Let $f : L \to L$ be a function over L. A set $X \subseteq L$ is called *f-compatible* iff for each $x \in X$, $f(x) \in X$.

Lemma 3.17. Let $\alpha \leq \kappa$, $f : L \to L$ and $\{X_\beta\}_{\beta<\alpha}$ be a sequence of f-compatible subsets of L. Then $\bigcap_{\beta<\alpha} X_\beta$ is f-compatible.

Proof. Suppose $x \in \bigcap_{\beta<\alpha} X_\beta$. For every $\beta < \alpha$, we have that $x \in X_\beta$. Since X_β is f-compatible, $f(x) \in X_\beta$. Since this holds for any $\beta < \alpha$, $f(x) \in \bigcap_{\beta<\alpha} X_\beta$. □

CHAPTER 4

THE FIXED POINT THEOREM

In this chapter we develop the main fixed point theorem.

Definition 4.1. A function $f : L \to L$ is called *stratified monotonic* iff it is $\sqsubseteq_\alpha$-monotonic for each $\alpha < \kappa$.

Theorem 4.2. Suppose that $f : L \to L$ is stratified monotonic and let $v \in L$ such that $v \sqsubseteq f(v)$. Then the sets $P_v = \{z \in L : (f(z) \sqsubseteq z) \wedge (v \sqsubseteq z)\}$ and $F_v = \{z \in L : (f(z) = z) \wedge (v \sqsubseteq z)\}$ have least elements that coincide.

Proof. Let us define for each ordinal $\alpha < \kappa$:

$$X_\alpha = \begin{cases} L, & \alpha = 0 \\ \{x \in Y_\beta : f(x) =_\beta x \wedge \forall y \in Y_\beta (f(y) \sqsubseteq_\beta y \to x \sqsubseteq_\beta y)\}, & \alpha = \beta + 1 \\ \bigcap_{\beta<\alpha} X_\beta, & \text{otherwise} \end{cases}$$

where

$$Y_\beta = \begin{cases} \{x \in X_\beta : v \sqsubseteq_\beta x\}, & v \in X_\beta \\ X_\beta, & \text{otherwise} \end{cases}$$

Suppose $\beta < \kappa$ and let $\alpha = \beta + 1$. We claim that, if there exists some x such that $x \in X_\alpha$, then $X_\alpha = X_\beta \cap [x]_\beta$. Suppose $y \in X_\alpha$. Obviously $y \in X_\beta$. Since $f(x) \sqsubseteq_\beta x$ and $f(y) \sqsubseteq_\beta y$, by the definition of X_α, $y =_\beta x$. Therefore, we have that $X_\alpha \subseteq X_\beta \cap [x]_\beta$. Suppose now $z \in X_\beta \cap [x]_\beta$. If $v \in X_\beta$, then $v \sqsubseteq_\beta x =_\beta z$, so that $y \in Y_\beta$. Also, we have that $z =_\beta x$, $x =_\beta f(x)$, and by the $\sqsubseteq_\beta$-monotonicity of f, $f(x) =_\beta f(z)$, so that $z =_\beta f(z)$. Moreover, if for some $y \in Y_\beta$, $f(y) \sqsubseteq_\beta y$, we have $x \sqsubseteq_\beta y$, so that $z \sqsubseteq_\beta y$. Thus $z \in X_\beta \cap [x]_\beta$, and therefore, we have that $X_\beta \cap [x]_\beta \subseteq X_\alpha$, so that $X_\alpha = X_\beta \cap [x]_\beta$.

Now, we are going to prove by transfinite induction that for every ordinal $\alpha < \kappa$:

- For each $\gamma < \alpha$, $X_\gamma \supseteq X_\alpha$.
- X_α is $\approx_\alpha$-closed.
- X_α is complete.
- X_α is f-compatible.

When $\alpha = 0$, the claims are trivial.

Suppose now that $\alpha = \beta + 1$. We have that $X_\alpha \subseteq Y_\beta \subseteq X_\beta$. By the induction hypothesis, for each $\gamma < \beta$, $X_\gamma \supseteq X_\beta$, so that $X_\gamma \supseteq X_\alpha$ for each $\gamma < \alpha$. Furthermore, by the induction hypothesis, X_β is $\approx_\beta$-closed, complete and f-compatible. By Lemma 3.13, $(X_\beta, \sqsubseteq_\beta)$ is a complete prelattice. If $v \in X_\beta$, by Lemma 2.6, $(\{x \in X_\beta : v \sqsubseteq_\beta x\}, \sqsubseteq_\beta)$ is a complete prelattice. In any case $(Y_\beta, \sqsubseteq_\beta)$ is a complete prelattice. Moreover, if $v \in X_\beta$, since X_β is f-compatible and by Lemma 3.5, $v \sqsubseteq_\beta f(v)$. Therefore, by the $\sqsubseteq_\beta$-monotonicity of f, for any $x \in X_\beta$, $v \sqsubseteq_\beta f(v) \sqsubseteq_\beta f(x)$. So, we can see f as a function from Y_β to Y_β. By Theorem 2.8, there exists some $x \in Y_\beta$ such that $f(x) =_\beta x$ and for each y such that $f(y) \sqsubseteq_\beta y$, $x \sqsubseteq_\beta y$. We have that $x \in X_\alpha$, so by the previous claim, $X_\alpha = X_\beta \cap [x]_\beta$. For any $y, z \in X_\alpha$, $y =_\beta x =_\beta z$. By Property 1, $y \approx_\beta z$, and since $y =_\beta z$, we have that $y \approx_\alpha z$, so that X_α is $\approx_\alpha$-closed. By Lemma 3.12, X_α is complete. Finally, suppose $z \in X_\alpha$. Since X_β is f-compatible, $f(z) \in X_\beta$. Also, since $z =_\beta x$, by the $\sqsubseteq_\beta$-monotonicity of f, $f(z) =_\beta f(x)$, and we have that $f(x) =_\beta x$, so that $f(z) \in [x]_\beta$. Thus $f(z) \in X_\alpha$.

Suppose now that α is a limit ordinal and our claims hold for all ordinals less than α. X_α is obviously well-defined. By definition, $X_\gamma \supseteq X_\alpha$ whenever $\gamma < \alpha$. Using Lemma 3.10, X_α is $\approx_\alpha$-closed. Using Lemma 3.14 and Lemma 3.15, X_α is complete. Finally, by Lemma 3.17, X_α is f-compatible.

Let us define $X_\kappa = \bigcap_{\alpha<\kappa} X_\alpha$. By Lemma 3.14, X_κ is non-empty. So suppose $x \in X_\kappa$. By Lemma 3.10 and Property 2, x is the one and only one element of X_κ. By Lemma 3.17, X_κ is f-compatible, so that $f(x) = x$.

Now, we claim that $v \sqsubseteq x$. If $v \in X_\alpha$ for all $\alpha < \kappa$, then $v = x$. Otherwise, let α be the least ordinal such that $v \notin X_\alpha$. By the definition of X_α when α is a limit ordinal, α can not be a limit ordinal, so there exists some ordinal β such that $\alpha = \beta + 1$ and $v \in X_\beta$. $x \in X_\alpha$, so we have that $v \sqsubseteq_\beta x$. Moreover, $X_\alpha = X_\beta \cap [x]_\beta$. So if $v =_\beta x$, then $v \in X_\alpha$ would hold. Thus, $v \sqsubset_\beta x$, so that $v \sqsubset x$. Therefore $x \in F_v$, and since $F_v \subseteq P_v$, $x \in P_v$.

Now we claim that if $z \in P_v$, then $x \sqsubseteq z$. If $z \in X_\alpha$ for all $\alpha < \kappa$, then $x = z$. Otherwise, let α be the least ordinal such that $z \notin X_\alpha$. By the definition of X_α when α is a limit ordinal, α can not be a limit ordinal, so there exists some ordinal β such that $\alpha = \beta + 1$ and $z \in X_\beta$. Because X_β is f-compatible, $f(z) \in X_\beta$. And since X_β is $\approx_\beta$-closed, by Lemma 3.5, $f(z) \sqsubseteq_\beta z$. If $v \in X_\beta$, then by Lemma 3.5, $v \sqsubseteq_\beta z$. In any case $z \in Y_\beta$. Since $x \in X_\alpha$, $x \sqsubseteq_\beta z$. Moreover, $X_\alpha = X_\beta \cap [x]_\beta$. So if $x =_\beta z$, then $z \in X_\alpha$ would hold. So $x \sqsubset_\beta z$, and thus $x \sqsubset z$. Therefore, x is the least element of P_v, and since $F_v \subseteq P_v$, it is also the least element of F_v. $\square$

CHAPTER 5

PRE-FIXED AND POST-FIXED POINTS

Definition 5.1. Let $(S, \leq)$ be a partial order, $f : S \to S$ and $x \in S$. x is called:

- a $\leq$-*post-fixed point* of f iff $x \leq f(x)$.
- a $\leq$-*pre-fixed point* of f iff $f(x) \leq x$.
- a $\leq$-*fixed point* of f iff $f(x) = x$.

In this chapter we demonstrate three more general fixed-point theorems, namely that the set of fixed-points, the set of postfixed-points and the set of prefixed-points of a stratified-monotonic function forms a complete lattice.

Lemma 5.2. (See [5, Theorem 2.31].) Let $(S, \leq)$ be a partial order. $(S, \leq)$ is a complete lattice iff S has a $\leq$-bottom element and every non-empty subset $X \subseteq S$ has a $\leq$-least upper bound in S.

Lemma 5.3. Let $\alpha < \kappa$, $X \subseteq L$ and $x \in X$. If for all $y \in X$ either $y =_\alpha x$ or $y \sqsubseteq x$, then $\bigsqcup X =_\alpha x$.

Proof. Let $X' = X \cap [x]_\alpha$. We have that $X' \subseteq [x]_\alpha$, so that X' is $\approx_{\alpha+1}$-closed. By Lemma 3.6, $\bigsqcup X' \approx_{\alpha+1} x$, so that $\bigsqcup X' \in [x]_\alpha$. For any $y \in X \setminus X'$, $y \sqsubseteq x \sqsubseteq \bigsqcup X'$. So $\bigsqcup X'$ is an $\sqsubseteq$-upper bound of X. Suppose z is an $\sqsubseteq$-upper bound of X. Then z is an $\sqsubseteq$-upper bound of X', so that $z \sqsubseteq \bigsqcup X'$. Thus, $\bigsqcup X = \bigsqcup X'$, and we have that $\bigsqcup X' =_\alpha x$. ☐

Lemma 5.4. Suppose that $f : L \to L$ is stratified monotonic and let $X \subseteq L$ such that for any $x \in X$, $x \sqsubseteq f(x)$. Then $\bigsqcup X \sqsubseteq f(\bigsqcup X)$.

Proof. If $\bigsqcup X \in X$ then $f(\bigsqcup X) = \bigsqcup X$ and we are done. Moreover, if $X = \emptyset$, $\bigsqcup \emptyset = \bigsqcap L \sqsubseteq f(\bigsqcap L)$. Assume therefore that $\bigsqcup X \notin X$ and $X \neq \emptyset$. We are going to show that $f(\bigsqcup X)$ is an $\sqsubseteq$-upper bound of X. Suppose $x \in X$. We have that $x \sqsubset_\alpha \bigsqcup X$ for some $\alpha < \kappa$. By the $\sqsubseteq_\alpha$-monotonicity of f, $f(x) \sqsubseteq_\alpha f(\bigsqcup X)$. If $f(x) \sqsubset_\alpha f(\bigsqcup X)$, then $x \sqsubseteq f(x) \sqsubset f(\bigsqcup X)$, so we are done. So, we can assume that $f(x) =_\alpha f(\bigsqcup X)$. We have that $x \sqsubseteq f(x)$. If $x \sqsubset_\beta f(x)$ for some $\beta \leq \alpha$, $f(x) =_\beta f(\bigsqcup X)$, so, by Lemma 3.4, $x \sqsubset_\beta f(\bigsqcup X)$, and thus $x \sqsubset f(\bigsqcup X)$, so we are done. Assume therefore that either $x = f(x)$ or $x \sqsubset_\beta f(x)$ for some $\beta > \alpha$. In either case, $x =_\alpha f(x)$, so that $x =_\alpha f(\bigsqcup X)$.

By Lemma 5.3, there exist some $y \in X$ such that $y \not\sqsubseteq x$ and $y \neq_\alpha x$. We have that $y \sqsubset_\beta \bigsqcup X$ for some $\beta < \kappa$. By the $\sqsubseteq_\beta$-monotonicity of f, $f(y) \sqsubseteq_\beta f(\bigsqcup X)$. Assume, for the sake of contradiction, that $\beta \leq \alpha$. If $\beta < \alpha$, by Property 1, $x =_\beta \bigsqcup X$, and by Lemma 3.4, $y \sqsubset_\beta x$ (contradiction). So, suppose that $\beta = \alpha$. We have that $y \sqsubseteq f(y)$. If $y \sqsubset_\gamma f(y)$ for some $\gamma < \alpha$, by Property 1 we have that $f(y) =_\gamma f(\bigsqcup X) =_\gamma x$, so by Lemma 3.4, $y \sqsubset_\gamma x$ (contradiction). Assume therefore that either $y = f(y)$ or $y \sqsubset_\gamma f(y)$ for some $\gamma \geq \alpha$. In either case, $y \sqsubseteq_\alpha f(y)$, and since $f(y) \sqsubseteq_\alpha f(\bigsqcup X)$ and $x =_\alpha f(\bigsqcup X)$, we have that $y \sqsubseteq_\alpha x$. Neither $y \sqsubset_\alpha x$ nor $y =_\alpha x$ can be true.

In any case we have a contradiction, so $\beta > \alpha$ must hold. Then by Property 1, $y =_\alpha \bigsqcup X$, and by Lemma 3.4, $x \sqsubset_\alpha y$. Also, by the $\sqsubseteq_\alpha$-monotonicity of f, $f(y) =_\alpha f(\bigsqcup X)$. We have that $y \sqsubseteq f(y)$. If $y \sqsubset_\gamma f(y)$ for some $\gamma < \alpha$, $x =_\gamma y \sqsubset_\gamma f(y) =_\gamma f(\bigsqcup X)$. If $y = f(y)$ or $y \sqsubset_\gamma f(y)$ for some $\gamma \geq \alpha$, then $y \sqsubseteq_\alpha f(y)$, so that $x \sqsubset_\alpha y \sqsubseteq_\alpha f(y) =_\alpha f(\bigsqcup X)$. In any case, by Lemma 3.4, $x \sqsubset f(\bigsqcup X)$. Therefore, $f(\bigsqcup X)$ is an $\sqsubseteq$-upper bound of X, so that $\bigsqcup X \sqsubseteq f(\bigsqcup X)$ □

Theorem 5.5. Suppose that $f : L \to L$ is stratified monotonic and let P be the set of post-fixed points of f. Then, $(P, \sqsubseteq)$ is a complete lattice.

Proof. By Lemma 5.2 it is sufficient to prove that P has a $\sqsubseteq$-bottom element and that every non-empty subset $X \subseteq P$ has a $\sqsubseteq$-least upper bound in P. Obviously $\bigsqcap L$ is the $\sqsubseteq$-bottom element of P. Let $X \subseteq P$. By Lemma 5.4, $\bigsqcup X \in P$, so that $\bigsqcup X$ is the $\sqsubseteq$-least upper bound of X. □

Also, by duality, we can also get the following theorem:

Theorem 5.6. Suppose that $f : L \to L$ is stratified monotonic and let P be the set of pre-fixed points of f. Then, $(P, \sqsubseteq)$ is a complete lattice.

Theorem 5.7. Suppose that $f : L \to L$ is stratified monotonic and let P be the set of fixed points of f. Then, $(P, \sqsubseteq)$ is a complete lattice.

Proof. Let $X \subseteq P$ be a set of fixed points of f. We show that X has a least upper bound in P. By a dual argument, one can also show that X has a greatest lower bound in P. These two statements suffice to establish that $(P, \sqsubseteq)$ is a complete lattice.

Obviously, X is also a set of post-fixed points of f. By Lemma 5.4, $\bigsqcup X$ is a post-fixed point of f. By Theorem 4.2, the set $\{x \in L : (f(x) = x) \wedge (\bigsqcup X \sqsubseteq x)\}$ has a least element, say z. This element z is the least fixed point of f that is an upper bound of all the elements of X. Therefore, z is the least upper bound of X in the partial order $(P, \sqsubseteq)$. This completes the proof of the theorem. □

CHAPTER 6

APPLICATIONS

In this chapter we present two applications of the fixed point theorem developed in this paper. The first is an application to the semantics of logic programs with negation and the second an application regarding sequences of (possibly) transfinite length.

6.1 The infinite-valued semantics

In [11] it is demonstrated that for every logic program P with negation, there exists a special model which can be taken as its intended semantics. It is also shown in [11] that this model is the least fixed point of the *immediate consequence operator* T_P of the program P. The proof of this result is given in [11] in a lengthy and somewhat ad-hoc way. In the following, we demonstrate that this result is a simple consequence of the theory developed in the present paper. To avoid an extensive presentation of the material in [11], we will introduce only the basic notions that are needed for this result to be established. The interested reader should consult [11] for additional details.

The basic notion that needs to be introduced, is that of an *infinite-valued interpretation*. Such interpretations are used in [11] to give meaning to logic programs with negation. Intuitively, an infinite-valued interpretation is a generalization of classical interpretations of logic programs [9] to an infinite-valued logic which contains one truth value F_α and one T_α for each countable ordinal α, together with a neutral truth value 0. Intuitively, F_0 and T_0 are the classical *False* and *True* values and 0 is the *undefined* value. The intuition behind the new values is that they express different levels of truth and falsity. Let V be this set of truth values, ie.,

$$V = \{F_\alpha : \alpha < \Omega\} \cup \{T_\alpha : \alpha < \Omega\} \cup \{0\}$$

where Ω is the first uncountable ordinal. We will need the following definition from [11]:

Definition 6.1. The *order* of a truth value is defined as follows: *order*$(T_\alpha) = \alpha$, *order*$(F_\alpha) = \alpha$ and *order*$(0) = \Omega$.

Let Z be a non-empty set of variables. Intuitively, the variables in Z are used to construct propositional logic programs in [11]. For the purposes of this section, it suffices to know that Z is just a non-empty set. Then:

Definition 6.2. An *infinite-valued interpretation* (or simply, *interpretation*) I is a function from the set Z to the set of truth values V.

Following [11], we define various relations on interpretations:

Definition 6.3. Let $I \in V^Z$ be an interpretation and let $v \in V$. Then $I \parallel v = \{z \in Z \mid I(z) = v\}$.

Definition 6.4. Let $I, J \in V^Z$ be interpretations and $\alpha < \Omega$. We write $I =_\alpha J$, if for all $\beta \leq \alpha$, $I \parallel T_\beta = J \parallel T_\beta$ and $I \parallel F_\beta = J \parallel F_\beta$.

Definition 6.5. Let $I, J \in V^Z$ be interpretations and $\alpha < \Omega$. We write $I \sqsubset_\alpha J$, if for all $\beta < \alpha$, $I =_\beta J$ and either $I \parallel T_\alpha \subset J \parallel T_\alpha$ and $I \parallel F_\alpha \supseteq J \parallel F_\alpha$, or $I \parallel T_\alpha \subseteq J \parallel T_\alpha$ and $I \parallel F_\alpha \supset J \parallel F_\alpha$. We write $I \sqsubseteq_\alpha J$ if $I =_\alpha J$ or $I \sqsubset_\alpha J$. We write $I \sqsubset J$ if there exists $\alpha < \Omega$ such that $I \sqsubset_\alpha J$. We write $I \sqsubseteq J$ if $I \sqsubset J$ or $I = J$.

We can now show that the set of infinite-valued interpretations together with the above relations, forms a lexicographic lattice structure.

Lemma 6.6. Let L be the set of infinite-valued interpretations. Let the relations $\sqsubseteq$ and $\{\sqsubseteq_\alpha\}_{\alpha<\Omega}$ be as in Definition 6.5. Then, the triple $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\Omega}\rangle$ is a lexicographic lattice structure.

Proof. The set L is a complete lattice (see [6]). Property 1 holds directly due to Definition 6.5. To verify that Property 2 holds, let I, J be infinite-valued interpretations and assume that for all $\alpha < \Omega$ it is $I =_\alpha J$. For any $\alpha < \Omega$ it follows by the definition of the $=_\alpha$ relation (Definition 6.4) that given an arbitrary $z \in Z$, $I(z) = T_\alpha$ iff $J(z) = T_\alpha$ and $I(z) = F_\alpha$ iff $J(z) = F_\alpha$; this implies that $I(z) = 0$ iff $J(z) = 0$. Therefore, for all $z \in Z$, $I(z) = J(z)$, ie., $I = J$. To verify Property 3, consider an arbitrary interpretation I and let $\alpha < \Omega$ be an ordinal. The set $[I]_\alpha$ has a $\sqsubseteq$-least element J defined as:

$$J(z) = \begin{cases} I(z) & \text{if } \mathit{order}(I(z)) \leq \alpha \\ F_{\alpha+1} & \text{otherwise} \end{cases}$$

and a $\sqsubseteq$-greatest element K defined as follows:

$$K(z) = \begin{cases} I(z) & \text{if } \mathit{order}(I(z)) \leq \alpha \\ T_{\alpha+1} & \text{otherwise} \end{cases}$$

It is straightforward to verify using Definition 6.5 that J and K are indeed the $\sqsubseteq$-least and $\sqsubseteq$-greatest elements of $[I]_\alpha$. □

In [11] an operator $T_P : L \to L$ is defined for every logic program P, where L is the set of infinite-valued interpretations. It is demonstrated that for every $\alpha < \Omega$, T_P is α-monotonic. Moreover, it is demonstrated through a lengthy reasoning, that T_P has a least fixed point (see Sections 6 and 7 in [11]), which is taken as the intended meaning of the program. This result can now be obtained in a much easier way as a direct consequence of the theory developed in this paper: since T_P is α-monotonic for all $\alpha < \Omega$, and since L is a lexicographic lattice structure, it follows from Theorem 4.2 that T_P has a least fixed point.

6.2 Transfinite sequences over complete lattices

In this section we consider (possibly transfinite) sequences over complete lattices. This is actually a generalization of ω-words discussed in Chapter 3. Sets of transfinite sequences over complete lattices have been studied by Henry Crapo in [4], where they are referred as *lexicographic lattices*. In this section we demonstrate that these sets induce, in a natural way, lexicographic lattice structures.

In the rest of this section we assume that κ is a fixed ordinal and $(Q, \leq)$ is a complete lattice. The set Q^κ of the functions from κ to Q can be viewed as sequences of length κ over Q. These sequences have an intuitive lexicographic order: suppose $f, g \in Q^\kappa$ such that $f \neq g$ and let α be the least ordinal such that $f(\alpha) \neq g(\alpha)$. Then $f \sqsubset g$ if $f(\alpha) < g(\alpha)$ and $g \sqsubset f$ if $g(\alpha) < f(\alpha)$.

Definition 6.7. Let $f, g \in Q^\kappa$. We define $f \sqsubseteq_\alpha g$ if $f(\beta) = g(\beta)$ for all $\beta < \alpha$ and $f(\alpha) \leq g(\alpha)$. We write $f =_\alpha g$ if $f \sqsubseteq_\alpha g$ and $g \sqsubseteq_\alpha f$. We write $f \sqsubset_\alpha g$ if $f \sqsubseteq_\alpha g$ but $f =_\alpha g$ does not hold. We write $f \sqsubset g$ if $f \sqsubset_\alpha g$ for some $\alpha < \kappa$. We write $f \sqsubseteq g$ if $f \sqsubset g$ or $f = g$.

The following lemmas hold:

Lemma 6.8. For each $\alpha < \kappa$, $\sqsubseteq_\alpha$ is a preorder.

Proof. Let $\alpha < \kappa$ be an ordinal and $f, g, h \in Q^\kappa$. We have that $f(\beta) = f(\beta)$ for all $\beta < \alpha$ and $f(\alpha) \leq f(\alpha)$ by the reflexivity of $\leq$, so that $f \sqsubseteq_\alpha f$. Suppose now that $f \sqsubseteq_\alpha g \sqsubseteq_\alpha h$. Then $f(\beta) = g(\beta) = h(\beta)$ for all $\beta < \alpha$ and $f(\beta) \leq g(\beta) \leq h(\beta)$. by the transitivity of $\leq$ and the equality, so that $f \sqsubseteq_\alpha h$. Thus $\sqsubseteq_\alpha$ is reflexive and transitive. □

Lemma 6.9. (See [4, Proposition 2].) $\sqsubseteq$ is a partial order.

Lemma 6.10. (See [4, Theorem 1].) $(Q^\kappa, \sqsubseteq)$ is a complete lattice.

Lemma 6.11. $\sqsubseteq$ is determined by the sequence $\{\sqsubseteq_\alpha\}_{\alpha<\kappa}$.

Proof. Let $f, g \in Q^\kappa$. By definition, $f \sqsubset g$ iff $f \sqsubset_\alpha g$ for some $\alpha < \kappa$. □

Definition 6.12. Let $\alpha > \kappa$ be an ordinal and $f \in Q^\kappa$. We define $f|_\alpha, f|^\alpha \in Q^\kappa$ such that:

$$f|_\alpha(\beta) = \begin{cases} f(\beta), & \beta \leq \alpha \\ \bot, & \text{otherwise} \end{cases}$$

$$f|^\alpha(\beta) = \begin{cases} f(\beta), & \beta \leq \alpha \\ \top, & \text{otherwise} \end{cases}$$

where $\bot$ is the least element and $\top$ is the greatest element of Q.

Lemma 6.13. The triple $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\kappa}\rangle$ is a lexicographic lattice structure.

Proof. Property 1 holds by definition. For Property 2, if for some $f, g \in Q^\kappa$, $f =_\alpha g$ for all $\alpha < \kappa$ then $f(\alpha) = g(\alpha)$ for all $\alpha < \kappa$, so that $f = g$. Finally, for Property 3, for any $f \in Q^\kappa$ and $\alpha < \kappa$, $f|_\alpha$ is the $\sqsubseteq$-least element and $f|^\alpha$ is the $\sqsubseteq$-greatest element of $[x]_\alpha$. □

We now derive a simple lemma for a class of functions over sequences that always have a least fixed point. A function $T : Q^\kappa \to Q^\kappa$ will be called *past-dependent* if, intuitively speaking, for every $f \in Q^\kappa$ and for every ordinal $\alpha < \kappa$, the value of the sequence $T(f)$ at index α depends only on the values that the sequence f has at ordinal indices that are strictly less than α. More formally:

Definition 6.14. The function $T : Q^\kappa \to Q^\kappa$ will be called *past-dependent* if the following condition holds: for each $\alpha < \kappa$ and for all $f, g \in Q^\kappa$, if $f(\beta) = g(\beta)$ for all $\beta < \alpha$ then $T(f)(\beta) = T(g)(\beta)$ for all $\beta \leq \alpha$.

We have the following simple lemma:

Lemma 6.15. Every past-dependent function is stratified monotonic.

Proof. Let $T : Q^\kappa \to Q^\kappa$ be a past-dependent function. We show that T is α-monotonic for each $\alpha < \kappa$. Suppose $\alpha < \kappa$ and let $f, g \in Q^\kappa$ such that $f \sqsubseteq_\alpha g$. By definition, $f(\beta) = g(\beta)$ for all $\beta < \alpha$. By hypothesis, $T(f)(\beta) = T(g)(\beta)$ for all $\beta \leq \alpha$, so that $T(f) \sqsubseteq_\alpha T(g)$. □

Corollary 6.16. Every past-dependent function has a least fixed point and a greatest fixed point.

Proof. Immediate by Theorem 4.2. □

Example 6.17. Let $\kappa = \omega$ and $Q = \{a, b\}$. $\{a, b\}^\omega$ are the infinite strings over alphabet $\{a, b\}$. Let $\leq$ be the usual ordering on $\{a, b\}$. Obviously, $(\{a, b\}, \leq)$ is a complete lattice. Using Lemma 6.13, $\langle L, \sqsubseteq, \{\sqsubseteq_\alpha\}_{\alpha<\omega}\rangle$ is a lexicographic lattice structure and $\sqsubseteq$ is the usual lexicographic ordering over strings.

Let's define $T : \{a, b\}^\omega \to \{a, b\}^\omega$ such that for all $f \in \{a, b\}^\omega$ and $n \in \omega$:

$$T(f)(n) = \begin{cases} b, & n = 0 \\ f(n-1), & n \text{ is odd} \\ f(n-2), & n \text{ is even and greater than } 0 \end{cases}$$

Although T is not monotonic under the lexicographic ordering, it follows by Lemma 6.15 that T is stratified monotonic. By Corollary 6.16, T has a least fixed point and a greatest fixed point. □

CHAPTER 7

CONCLUSIONS AND FUTURE WORK

We have presented a novel, non-constructive proof of the fixed point theorem proposed by A. Charalambidis, G. Chatziagapis and P. Rondogiannis [1]. The initial proof of the theorem, as presented at LICS 2020, is constructive. Even though the constructive proofs are useful is some applications, non-constructive proofs often give a better intuition of the results and highlight some hidden properties. Using the novel approach that we developed, we proved that the set of fixed points of functions over lexicographic lattice structures form a complete lattice, which was the main unresolved open question of [1]. This result makes the analogy with the Knaster-Tarski theorem even firmer. Finally, we have formally verified our proofs through the Coq proof assistant. The code can be retrieved from `https://github.com/giannosch/lexicographic-fixed-point`.

Most of the lexicographic lattice structures can be described as sequences over complete lattices. This observation is important in our novel approach. It is interesting to investigate whether we can give an alternative definition to lexicographic lattice structures based on such sequences.

Moreover, classical logic programming semantics are closely related with classical, two-valued set theory, in the sense that for each model an atom is either an element of the model, so it is true, or not, so it is false. Since logic programming with negation can be described with infinite valued models, it may give rise to a novel, non-standard, infinite-valued set theory. Rondogiannis and Wadge in [11] even defined some notion of intersection between models in order to prove a theorem similar to the model intersection theorem is classical logic programming. It is interesting to explore the properties of such set theory.

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