Structural and Topological Graph Theory and Well-Quasi-Ordering

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Μεταπτυχιακό Πρόγραμμα «Αλγόριθμοι, Λογική και Διακριτά Μαθηματικά» – 2016

Η παρούσα Διπλωματική Εργασία
εκπονήθηκε στα πλαίσια των σπουδών
για την απόκτηση του
Μεταπτυχιακού Διπλώματος Ειδίκευσης
«Αλγόριθμοι, Λογική και Διακριτά Μαθηματικά»
που απονέμει το
Τμήμα Πληροφορικής και Τηλεπικοινωνιών
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2.	....	....
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Για την ισότητα, τις παρενθέσεις, την έννοια της μεταβλητής, τους αρνητικούς αριθμούς, την εντολή if-then-else, γιατί «τα πάντα είναι μαθηματικά και τίποτα δεν είναι μόνο μαθηματικά»,

αφιερώνω την εργασία αυτή στον πρώτο μου Δάσκαλο μαθηματικών, προγραμματισμού και πολλών άλλων, τον Πατέρα μου.

ABSTRACT

In their Graph Minors series, Neil Robertson and Paul Seymour among other great results proved Wagner's conjecture which is today known as the "Robertson and Seymour's theorem". In every step along their way to the final proof, each special case of the conjecture which they were proving was a consequence of a "structure theorem", that sufficiently general graphs contain minors or other sub-objects that are useful for the proof - or equivalently, that graphs that do not contain a useful minor have a certain restricted structure, deducing that way also a useful information for the proof. The main object of this thesis is the presentation of -relatively short- proofs of several Robertson and Seymour's theorem's special cases, illustrating by this way the interplay between structural graph theory and graphs' well-quasi-ordering. We also present the proof of perhaps the most important special case of Robertson and Seymour's theorem which states that embeddability in any fixed surface can be characterized by forbidding finitely many minors. The latter result is deduced as a well-quasi-ordering result, indicating by this way the interplay between topological graph theory and well-quasi-ordering theory. Finally, we survey results regarding the well-quasi-ordering of graphs by other than the minor graphs' relations.

ΣΥΝΟΨΗ

Στη σειρά εργασιών Ελασσόνων Γραφημάτων, οι Neil Robertson και Paul Seymour μεταξύ άλλων σπουδαίων αποτελεσμάτων, απέδειξαν την εικασία του Wagner που σήμερα είναι γνωστή ως το Θεώρημα των Robertson και Seymour. Σε κάθε τους βήμα προς την συναγωγή της τελικής απόδειξης της εικασίας, κάθε ειδική περίπτωση αυτής που αποδείκνυαν ήταν συνέπεια ενός "δομικού θεωρήματος" το οποίο σε γενικές γραμμές ισχυριζόταν ότι ικανοποιητικά γενικά γραφήματα περιέχουν ως ελάσσονα γραφήματα ή άλλες δομές που είναι χρήσιμα για την απόδειξη, ή ισοδύναμα, ότι η δομή των γραφημάτων τα οποία δεν περιέχουν ένα χρήσιμο για την απόδειξη γράφημα ως έλασσον είναι κατά κάποιο τρόπο περιορισμένη συνάγοντας έτσι και πάλι μια χρήσιμη πληροφορία για την απόδειξη. Στην παρούσα εργασία, παρουσιάζουμε -σχετικά μικρές- αποδείξεις διαφόρων ειδικών περιπτώσεων του Θεωρήματος των Robertson και Seymour, αναδεικνύοντας με αυτό τον τρόπο την αλληλεπίδραση της δομικής θεωρίας γραφημάτων με την θεωρία των καλών-σχεδόν-διατάξεων. Παρουσιάζουμε ακόμα την ίσως πιο ενδιαφέρουσα ειδική περίπτωση του Θεωρήματος των Robertson και Seymour, η οποία ισχυρίζεται ότι η εμβαπτισιμότητα σε κάθε συγκεκριμένη επιφάνεια δύναται να χαρακτηριστεί μέσω της απαγόρευσης πεπερασμένων το πλήθος γραφημάτων ως ελάσσονα. Το τελευταίο αποτέλεσμα συνάγεται ως ένα αποτέλεσμα της θεωρίας των καλών-σχεδόν-διατάξεων αναδεικνύοντας με αυτό τον τρόπο την αλληλεπίδρασή της με την τοπολογική θεωρία γραφημάτων. Τέλος, σταχυολογούμε αποτελέσματα αναφορικά με την καλή-σχεδόν-διάταξη κλάσεων γραφημάτων από άλλες -πέραν της σχέσης έλασσον- σχέσεις γραφημάτων.

ACKNOWLEDGMENTS

First and foremost I would like to express my deep gratitude to my advisor, Professor Dimitrios M. Thilikos for introducing me via the lesson «Graph Theory» to Graph Minors, for his guidance, encouragement, patience, and helpful critique throughout all the stages of this work, but most of all for his general contribution via the way that his approaching any subject of mathematics that he is teaching and via his strong work ethic.

I would also like to express my sincere thanks to the other two members of my thesis examination committee, Assistant Professor Archontia C. Giannopoulou and Professor Stavros G. Kolliopoulos, for their time and patience.

I am also thankful to the inter-institutional Graduate Program «Algorithms, Logic and Discrete Mathematics», especially I would like to thank its director Professor Panagiotis Rondogiannis for his excellent teaching which I enjoyed in the context of the lesson «Semantics of Programming Languages», but at most, for all the valuable discussions that we had (for scientific and other issues) and for the help that I got via those discussions in an important stage of my course in my master studies. I would like also to thank Dr. Dimitris Zoros for his very good work in the lessons «Recursion Theory», «Computational Complexity» and «Parametrized Complexity and Algorithms»and for being always willing to help.

Moreover, I would like to thank the Department of Mathematics of NKUA and especially the part of its academic staff which despite any adverse conditions that (may) exist, selflessly offers the best, especially I would like to express my deep gratitude and admiration to Professor Apostolos Giannopoulos.

My friend and colleague Στέλιος-Εριόν Μπότσι -especially via his paradigm- gave me very important help at my first "serious" steps in mathematics during my undergraduate years, so i would also like to express my gratitude to him.

I should not neglect to give credit to two websites -whose name I cannot remember right now (!)- whose help in my research were very valuable.

In the context of this little and public text I can only say «*Ευχαριστώ*» to each of the following: Αλεξία Καραντάνα, Βασιλίνα Μπισμπίκη, Εμμανουέλα Μπαλαγιάννη, Θωμά Κατσιούλα, Κώστα Κατσικέα, Ματίνα Κόβε, Νίκο Καραγίαννη Αξυπολιτίδη, Παναγιώτη Ρεϊτζόπουλο.

Finally, it wouldn't be a cliché to say that there no words to thank my Parents.

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CHAPTER 1

BASIC DEFINITIONS AND NOTATIONS

In this chapter we give the basic definitions needed for the understanding of the concepts which we discuss in further chapters and we fix the notation we plan to use throughout this thesis. Some terms which can be better understood in their proper setting will be introduced there. The only knowledge assumed here is a familiarity with simple sentences of propositional and first-order logic and with trivial set theory. We will systematically make use of the following logic and set-theoretic symbols: $\rightarrow, \Rightarrow, \Leftrightarrow, \neg, \wedge, \vee, \forall, \exists, \nexists, \exists!, :=, =:, \emptyset, \in, \notin, \subseteq, \subsetneq, \supseteq, \supsetneq, \cup, \cap$. The semantics of the these symbols will be the usual[1].

1.1 Sets, relations and functions

For a rigourous introduction in set theory and logic, we refer the interested reader to [93] and [39] respectively.

Definition 1.1.1 (partition)**.** Given a set A, a set P will be said to be a *partition* of A if and only if the following hold:

(i) $(\forall X \in P)[X \neq \emptyset]$;

(ii) $\bigcup_{X \in P} X = A$;

(iii) $(\forall X_1, X_2 \in P)[X_1 \cap X_2 = \emptyset]$.

Definition 1.1.2 (cardinality of a set\order of a set[2])**.** Given a set A the *cardinality* of A -denoted by $|A|$- is the number of the elements of A. We also call the cardinality of a set as its *order*.

Notation 1.1.3 (natural, integer, rational and real numbers)**.** We denote the sets of natural, integer, rational and real numbers by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ respectively.

Definition 1.1.4 (powerset of a set)**.** Let X be a set, the *powerset* of X -denoted by $\mathcal{P}(X)$- is the set of all subsets of X, that is $\mathcal{P}(X) = \{A | A \subseteq X\}$

[1] see e.g. [39, 93]

[2] When we define a new notion we use the symbol "\" between different equivalent names of this notion.

Definition 1.1.5 (k-subset, k-subsets)**.** Let X be a set and k be a positive integer. A *k-subset* of X, is a subset of X which contains exactly k elements. The *k-subsets* of X -denoted by $[X]^k$- is the set of all subsets of X with order k, that is $[X]^k = \{A|(A \subseteq X) \wedge (|A| = k)\}$.

Notation 1.1.6 (set of finite subsets)**.** Given a set X, we denote by $[X]^{<\omega}$ the set of all finite subsets of X.

Definition 1.1.7 (Cartesian Product of sets)**.** Let $n \geq 2$ be a natural number and $X_1, \ldots X_n$ be sets. The *Cartesian product* of the sets $X_1, \ldots X_n$ -denoted by $X_1 \times, \ldots, \times X_n$ and by $\Pi_{i=1}^n X_i$- is the set: $\{(x_1, \ldots, x_n)|(x_1 \in X_1) \wedge \ldots \wedge (x_n \in X_n)\}$.

Definition 1.1.8 (n-ary Cartesian power of a set)**.** Let $n \geq 2$ be a natural number and X be a set, then the Cartesian product $\underbrace{X \times \ldots \times X}_{n \text{ times}}$ is said to be the *n-ary Cartesian power* of the set X.

We proceed with definitions relative with binary relations, the main binary relations that we will consider throughout this thesis are the *well-quasi-orders* which we define in Chapter 2 (Definition 2.1.2).

Definition 1.1.9 (binary relation)**.** Let A, B be a sets, a *binary relation* over the sets A, B is a subset of $A \times B$. A binary relation on A is a subset of $A \times A$.

Notation 1.1.10. Given a binary relation, say R, we denote by xRy the fact that $(x, y) \in R$.

Definition 1.1.11 (reflexive/irreflexive/symmetric/antisymmetric/transitive relation)**.** Let X be a set and R be a binary relation on X. The relation R will be said to be:

- *reflexive* if and only if $(\forall x \in X)[xRx]$;
- *irreflexive* if and only if $(\forall x, y \in X)[xRy \to x \neq y]$;
- *symmetric* if and only if $(\forall x, y \in X)[xRy \to yRx]$;
- *antisymmetric* if and only if $(\forall x, y \in X)[(xRy) \wedge (x \neq y) \to \neg(yRx)]$;
- *transitive* if and only if $(\forall x, y, z \in X)[(xRy) \wedge (yRz) \to xRz]$.

Definition 1.1.12 (quasi-order)**.** Let X be a set and R be a binary relation on X. The relation R will be said to be a *quasi-order* if and only if is reflexive and transitive. If R is a quasi-order on X, we will say that the set X is quasi-ordered by the relation R.

Definition 1.1.13 (extension of a quasi-order)**.** Given a set X and two quasi-orders R_1, R_2 on X. We say that the relation R_2 is an *extension* of R_1 if and only if $R_1 \subseteq R_2$. In this case, we also say that R_1 is a restriction of R_2 or that R_2 extends R_1.

Definition 1.1.14 (partial order)**.** Let X be a set and R be a binary relation on X. The relation R will be said to be a *partial order* if and only if is reflexive, transitive and antisymmetric. If R is a partial order on X, we will say that the set X is partially ordered by the relation R.

Observation 1.1.15. It follows immediate from the above definitions that every partial order is a quasi-order.

Definition 1.1.16 (strict partial order)**.** Let X be a set and R be a binary relation on X. The relation R will be said to be a *strict partial order* if and only if is irreflexive and transitive. If R is a strict partial order on X, we will say that the set X is strictly partially ordered by the relation R.

Observation 1.1.17. It follows immediate from the above definition that every strict partial order is antisymmetric.

Definition 1.1.18 (equivalence relation)**.** Let X be a set and R be a binary relation on X. The relation R will be said to be an *equivalence binary relation* if and only if is reflexive, symmetric and transitive.

Definition 1.1.19 (classes of equivalence, representative of a class of equivalence)**.** Let X be a set and R be an equivalence relation on X. For every $x_0 \in X$ the set $\{x \in X | x \sim x_0\}$ is the *class of equivalence* of x_0. Any element of a class of equivalence is said to be a *representative of a class of equivalence*.

Definition 1.1.20 (chain)**.** Let X be a set, R be a binary relation on X and let also A be a subset of X. The set A will be said to be a *chain* -w.r.t. R- if and only if $(\forall x, y \in A)[(xRy) \vee (yRx)]$.

Definition 1.1.21 (incomparable elements)**.** Let R be a binary relation on a set X, and $x, y \in X$. We say that x and y are *incomparable* -w.r.t R- elements of X if and only if nor yRx neither xRy. If x, y are incomparable elements of a set, we denote that by $x|y$.

Definition 1.1.22 (antichain)**.** Let X be a set, R be a binary relation on X and let also A be a subset of X. The set A will be said to be an *antichain* -w.r.t. R- if and only if $(\forall x, y \in A)[x \neq y \Rightarrow x|y\,]$.

Comment 1.1.23. *We often use the symbol $\leq$ to denote a binary relation. In that case we denote the fact that $(x, y) \notin \leq$ by $x \nleq y$ and the fact that $(x, y) \in \leq$ by $x \leq y$.*

Definition 1.1.24 (equivalent/minimum/minimal/maximum/maximal element w.r.t a quasi-order)**.** Let X be a nonempty set which is quasi ordered by a binary relation $\leq$. Two elements of X, say x_1, x_2 will be said to be *equivalent* (w.r.t $\leq$) if and only if $(x_2 \leq x_1) \wedge (x_1 \leq x_2)$. An element $x_1 \in A$ will be said to be:

- A *minimal* element of A (w.r.t. $\leq$) if and only if $(\forall x \in A)[x \neq x_1 \Rightarrow x \nleq x_1]$;
- a *minimum* element of A (w.r.t. $\leq$) if and only if $(\forall x \in A)[x_1 \leq x]$;
- a *maximal* element of A (w.r.t. $\leq$) if and only if $(\forall x \in A)[x \neq x_1 \Rightarrow x_1 \nleq x]$;
- a *maximum* element of A (w.r.t. $\leq$) if and only if $(\forall x \in A)[x \leq x_1]$.

When it is clear from the context the quasi-order to which we refer we shall omit the reference "w.r.t" in the above characterizations.

Definition 1.1.25 (function\map, domain of function, codomain of function)**.** Let A, B be two sets, a binary relation $f \subseteq A \times B$ will be said to be a *function* if and only if the following condition holds:

$$(\forall x \in A)[\big((x, y_1) \in f\big) \wedge \big((x, y_2) \in f\big) \rightarrow y_1 = y_2].$$

If $f \subseteq A \times B$ is a function, we denote that by $f : A \rightarrow B$, and we call the set A the *domain* of f and the set B the *codomain* of f we also denote by $f(x) = y$ the fact that $(x, y) \in f$.

Binary relation	Reflexive	Irreflexive	Symmetric	Antisymmetric	Transitive
Quasi-order	•				•
Partial order	•			•	•
Strict partial order		•		•	•
Equivalence binary relation	•		•		•

Table 1.1: Binary relations.

Definition 1.1.26 (restriction of a function)**.** Let $f : A \to B$ be a function and $A' \subseteq A$. Then the *restriction* of f in A' -denoted by $f|_{A'}$- is the function $f|_{A'} := \{(x, y) \in f | x \in A'\}$.

Definition 1.1.27 (partial function)**.** Let A, B be two sets, a binary relation $f \subseteq A \times B$ will be said to be a *partial function* if and only if there exists a subset $A' \subseteq A$ such that $f \subseteq A' \times B$ is a function with its domain to be the set A'.

Definition 1.1.28 (image of a set via a function, image of a function)**.** Given a function $f : A \to B$ and a set $C \subseteq A$ the set $\{y \in B | (\exists x \in C)[f(x) = y]\}$ will be called the *image of C via f*[3]. and will be denoted by $f(C)$, if $C = A$ then we call the set $f(A)$ the *image of f*.

Definition 1.1.29 (injection\one-to-one function)**.** Given a function $f : A \to B$, f will be said to be an *injection* or an *injective function* or a *one-to-one function* if and only if $(\forall x, y \in A)[f(x) = f(y) \to x = y]$.

Definition 1.1.30 (surjection\onto function)**.** Given a function $f : A \to B$, f will be said to be an *surjection* or an *surjective function* or an *onto function* if and only if $(\forall y \in B)(\exists x \in A)[f(x) = y]$.

Definition 1.1.31 (bijection\bijective function)**.** A function f will be said to be a *bijection* or a *bijective function* if and only if is injective and surjective.

Definition 1.1.32 (inverse function)**.** Given a one-to-one function $f : A \to B$, the function $\{(y, x) | (y \in B) \wedge (x \in A) \wedge (y = f(x))\} \subseteq B \times A$ will be said to be the *inverse* function of the function f and will be denoted by f^{-1}.

Definition 1.1.33 (composition of functions)**.** Given two functions $f : A \to B$ and $g : B \to C$, the function $\{(x, z) | (x \in A) \wedge (z \in C) \wedge (z = g(f(x)))\} \subseteq A \times C$ will be said to be the *composition* of f with g and will be denoted by $f \circ g$.

Definition 1.1.34 (Cartesian Product of arbitrary many sets)**.** Let I be an arbitrary set and $\{X_i | i \in I\}$ be a family of sets, then the set

$$\{x : I \to \bigcup_{i \in I} X_i | (\forall i \in I)[x(i) \in X_i]\}$$

will be said to be the *Cartesian product* of the sets $\{X_i | i \in I\}$ and will be denoted by $\Pi_{i \in I} X_i$.

Definition 1.1.35 (order homomorphism)**.** Let X, Y be two sets quasi-ordered by the relations $\leq_1$ and $\leq_2$ respectively. A function $f : X \to Y$ will be said to be an *order homomorphism* if and only if $(\forall x_1, x_2 \in X)[x_1 \leq_1 x_2 \to f(x_1) \leq_2 f(x_2)]$.

[3] When it is clear from the context to which function we refer, we shall omit the reference "via f".

Definition 1.1.36 (sequence, sequence on a set)**.** A function is said to be a *sequence* if and only if its domain coincidence with the set of natural numbers. Given a non-empty set X a *sequence on* X is a sequence whose image is subset of X.

Definition 1.1.37 (term, n-th term, set of terms of a sequence)**.** Given a sequence x on a set X, an element of X is called a *term* of x if and only if it 's an element of the image of x. The image of x is also called the *set of terms* of x. The *n-th term* of x is the element X which equals with $x(n)$ and which will be also denoted by x_n.

Notation 1.1.38. A sequence x will be also denoted as $(x_n)_{n\in\mathbb{N}}$.

Definition 1.1.39 (antichain sequence)**.** Given a set X, a binary relation R on X and a sequence $(x_n)_{n\in\mathbb{N}}$ on X, $(x_n)_{n\in\mathbb{N}}$ will be said to be an *antichain sequence* or simply *antichain* -w.r.t R- if and only if the set of its terms is an antichain w.r.t. R.

Definition 1.1.40 ((strictly) increasing/(strictly) decreasing)**.** Given a set X, a quasi-order R on X and a sequence $(x_n)_{n\in\mathbb{N}}$ on X, $(x_n)_{n\in\mathbb{N}}$ will be said to be a *increasing* sequence if and only if $(\forall i \in \mathbb{N})[x_i R x_{i+1}]$, if moreover $(\forall i \in \mathbb{N})[\neg(x_{i+1} R x_i)]$ then $(x_n)_{n\in\mathbb{N}}$ will be said to be *strictly increasing*. Analogously $(x_n)_{n\in\mathbb{N}}$ will be said to be an *decreasing* sequence if and only if $(\forall i \in \mathbb{N})[x_{i+1} R x_i]$, if moreover $(\forall i \in \mathbb{N})[\neg(x_i R x_{i+1})]$ then $(x_n)_{n\in\mathbb{N}}$ will be said to be *strictly increasing*.

Definition 1.1.41 (well-founded binary relation)**.** A binary relation $\leq$ on a set X is called *well-founded* on X if and only if every non-empty subset of X has a minimal (w.r.t $\leq$) element, that is: $(\forall A \subseteq X)[A \neq \emptyset \Rightarrow (\exists y \in A)(\forall x \in A)[\neg(x \leq y)]]$. Equivalently, if and only if it contains no countable infinite strictly decreasing chains: that is, there is no infinite sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X such that $(\forall n \in \mathbb{N})[x_{n+1} < x_n]$.

1.2 Graphs

This section consists a brief, self-contained introduction to graph theory. Although we do not follow in every notion the same notation, the main source for this section -which we also suggest as a further reading- was the textbook of Diestel [30] in graph theory.

Definition 1.2.1 (graph, vertices and edges of a graph, vertex set and edge set of a graph)**.** A *graph* is an ordered pair of two sets such that the second element of this pair is subset of the 2-subsets of the first element, that is, if G is a graph such that $G = (V, E)$, then $E \subseteq [V]^2$. The elements of the first set of the ordered pair are called *vertices* of the graph and those of the second set are called *edges* of the graph. That way the first set of the ordered pair will be said to be the *vertex set* or the *set of vertices* of the graph and the second will be said to be *edge set* of the graph. A graph with vertex set X will be said to be *a graph on* X. That way, if we say that G is a graph on $\mathbb{N}$ we mean that $V(G) = \mathbb{N}$

The usual way to illustrate a graph -which we also follow throughout this thesis- is by drawing a dot for each vertex of the graph and joining two of these dots by a line if the corresponding two vertices form an edge.

Definition 1.2.2 (empty graph)**.** The empty graph -denoted by $\emptyset$- is the graph $(\emptyset, \emptyset)$.

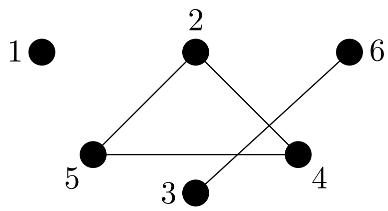

Figure 1.2.1: An illustration of the graph $\big(\{1,2,3,4,5,6\},\{\{3,6\},\{5,4\},\{5,2\},\{2,4\}\}\big)$. The vertex 1 is an isolated vertex of the graph.

Definition 1.2.3 (order of a graph, finite/infinite/countable/uncountable graph, trivial graph)**.** The *order* of a graph G is the cardinality of its vertex set. Graphs could be *finite, infinite, countable* or *uncountable* according to their order. A graph of order 0 or 1 will be said to be a *trivial graph*.

Notation 1.2.4. Let H be a graph. The vertex set and the edge set of H will be denoted by $V(H)$ and $E(H)$ respectively. This notation is independent of the specific name of those sets and is depended only on the name of the graph.

Definition 1.2.5 (union, intersection, disjoint graphs)**.** Let G, G' be two graphs. Then their *union* -denoted by $G \cup G'$- is the graph $(V(G) \cup V(G'), E(G) \cup E(G'))$, analogously their *intersection* -denoted by $G \cap G'$-is the graph $(V(G) \cap V(G'), E(G) \cap E(G'))$. The graphs G, G' will be said to be disjoint if and only if $G \cap G' = \emptyset$.

Notation 1.2.6. Given two graphs, say G, G', we denote by $G \backslash G'$ the graph $(V(G) \backslash V(G'), E(G) \backslash E(G'))$.

Definition 1.2.7 (complement of a graph)**.** Let G be a graph, then the *complement* of G -denoted by $\bar{G}$- is the graph $(V(G), [V]^2 \setminus E(G))$.

Definition 1.2.8 (incident vertex with an edge, endpoints/ends/endvertices of an edge)**.** Let G be a graph. A vertex $v \in V(G)$ will be said to be *incident* with an edge $e \in E(G)$ if and only if $v \in e$. Given an edge $e \in E(G)$, two points that are incident to e are the *endpoints* of e. The endpoints of an edge are also called *ends* and *endvertices* of this edge.

Definition 1.2.9 (adjacent vertices\neighbors, non-adjacent vertices)**.** Let G be a graph. Two vertices v, w of G will be said to be *adjacent* or *neighbours* if and only if $\{v, w\} \in E(G)$. Two vertices which are not adjacent will be said to be *non-adjacent*.

Definition 1.2.10 (the set of neighbors/ the neighborhood of a vertex)**.** Let G be a graph and $v \in V(G)$. The set of all vertices which are adjacent with of v will be called the *set of neighbors of* v or the *neighborhood of* v and will be denoted by $N_G(v)$.

Definition 1.2.11 (adjacent edges)**.** Let G be a graph, two edges of G, say e, f such that $e \neq f$ will be said to be *adjacent* if and only if they have one endpoint in common, that is, if and only if $|e \cap f| = 1$.

Definition 1.2.12 (degree\valency of a vertex, isolated vertex)**.** Let G be a graph and $v \in V(G)$. The *degree* of v -denoted by $deg(v)$- is the number of edges that v is incident with or equivalently the number $|N_G(v)|$. An *isolated* vertex is a vertex which has degree 0.

Definition 1.2.13 (minimum and maximum degree of a graph)**.** Given a graph G the *minimum degree* of G -denoted by $\delta(G)$- is the minimum degree of a vertex of this graph, that is, $\delta(G) = \min\{|N_G(v)|\ |v \in V(G)\}$. Analogously the *maximum degree* of G -denoted by $\Delta(G)$- is the maximum degree of a vertex of this graph, that is, $\Delta(G) = \max\{|N_G(v)|\ |v \in V(G)\}$.

Definition 1.2.14 (k-regular/regular graph)**.** If all the vertices of a graph have the same degree, say k, then the graph is called *k-regular*, or simply *regular*.

Definition 1.2.15 (cubic graph)**.** A 3-regular graph will be said to be a *cubic* graph.

Definition 1.2.16 (subcubic graph)**.** A graph G will be said to be *subcubic* if and only if $\Delta(G) \leq 3$.

Definition 1.2.17 (homomorphism and isomorphism between graphs)**.** Let G, G' be two graphs. A function $\phi : V(G) \to V(G')$ will be said to be a *homomorphism* from G to G' if and only if it preserves the adjacency of vertices, that is, $(\forall v, u \in V(G))[\{v, u\} \in E(G) \Rightarrow \{\phi(v), \phi(u)\} \in E(G')]$. If moreover, ϕ is bijective and its inverse function is also a homomorphism, then ϕ will be said to be an *isomorphism*.

Definition 1.2.18 (isomorphic graphs)**.** Two graphs, say G, G', will be said to be *isomorphic* -denoted by $G \simeq G'$ -if and only if there exist an isomorphism from $V(G)$ to $V(G')$.

Observation 1.2.19. The relation $\simeq$, as easily can be checked, is an equivalent relation. Hence a specific graph may be isomorphic with infinitely many graphs which belong to the same class of equivalence, we often use a representative of a class to refer any of its members.

Definition 1.2.20 (graph property)**.** A set of graphs $\mathcal{Q}$ will be said to be a *graph property* if and only if it is closed under isomorphism, that is, if and only if

$$(\forall G, G')[(G \simeq G') \wedge (G' \in \mathcal{Q}) \Rightarrow G \in \mathcal{Q}].$$

Definition 1.2.21 (graph invariant)**.** A function which takes graphs as arguments is called *graph invariant* if and only of it maps isomorphic graphs to same values.

Definition 1.2.22 (complete graph\clique)**.** For each positive natural number n, the graph K_n is defined to be the following:

$$\big(\{v_1, \ldots, v_n\}, \{\{v_i, v_j\}\ |1 \leq i < j \leq n\}\big).$$

A graph G on n vertices will be said to be *complete*, or a *complete graph on n vertices*, or a K_n or a *clique* if and only if $G \simeq K_n$.

Definition 1.2.23 (triangle)**.** A graph will be said to be a *triangle* if and only if it is a K_3.

Definition 1.2.24 (path, path of length n, trivial path)**.** For each natural number n, the graph P_n is defined to be the following:

$$\big(\{0, \ldots, v_n\}, \{\{v_i, v_{i+1}\}\ |0 \leq i \leq n-1\}\big).$$

A graph G on n vertices will be said to be a *path of length n* or a P_n if and only if $G \simeq P_n$, a *path* is any graph for which there exist a natural number n such that this graph is a path of length n. In the case in which $n = 0$, the path P_0 will be said to be a *trivial* path.

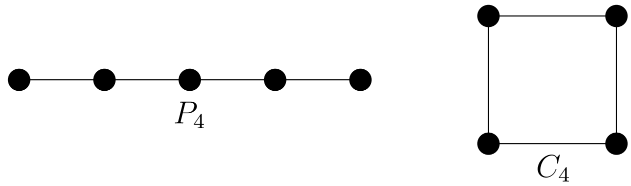

Figure 1.2.2: On the left hand side it is illustrated a path of length 4 and on the right hand side a cycle of length 4.

Notation 1.2.25. We often refer to a path by the natural sequence of its vertices, that is, given a path of length n, say,

$$P = \big(\{0, \dots, v_n\}, \{\{v_i, v_{i+1}\} \mid 0 \leq i \leq n-1\}\big).$$

we may also denote P by $(v_0, \dots v_n)$.

Comment 1.2.26. *Throughout this thesis sometimes we enumerate a set of paths and we use the symbol P_n to denote its n-th element, or we use this symbol to refer to another path. It will be clear from the context when we do so.*

Definition 1.2.27 (endpoints\ends of a path, vertices linked by a path)**.** Let $n \geq 2$, given a path P of length n the vertices of P which correspond (via an isomorphism) to the vertices v_0, v_n of P_n, will be said to be the *endpoints* or the *ends* of P, and will be also said to be *linked* by P.

Notation 1.2.28. Let P be a path and let v_0, v_n be its endpoints. We denote by $\mathring{P}$ the set $\big(V(P) \cup E(P)\big) \setminus \{v_0, v_n\}$.

Definition 1.2.29 (internally vertex-disjoint paths)**.** Two paths, say P_1, P_2 will be said to be *internally vertex-disjoint* if and only if $\mathring{P}_1 \cap \mathring{P}_2 = \emptyset$.

Definition 1.2.30 ((A, B)-path)**.** Let G be a graph and let $A, B \subseteq V(G)$. A path whose vertex set and the edge set are subsets of $V(G), E(G)$ respectively, and who has its one endpoint in A and its other endpoint in B will be said to be an *(A, B)-path*.

Definition 1.2.31 (walk, closed walk)**.** Let k be a positive natural number and G be a graph. An alternating sequence of vertices and edges of G, say $v_0, e_0, v_1, e_1 \dots, e_{k-1}, v_k$, will be said to be a *walk* of length k in G if and only if $(\forall i < k)[e_i = \{v_i, v_{i+1}\}]$. If moreover $v_0 = v_k$ the walk will be said to be *closed*.

Definition 1.2.32 (cycle, cycle of length n)**.** For each natural number $n \geq 3$, the graph C_n is defined to be the following:

$$\{\{v_1, \dots, v_n\}, \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq n-1\} \cup \{v_n, v_1\}\}.$$

A graph G on n vertices will be said to be a *cycle of length* n if and only if $G \simeq C_n$, a *cycle* is any graph for which there exist a natural number n such that this graph is a cycle of length n.

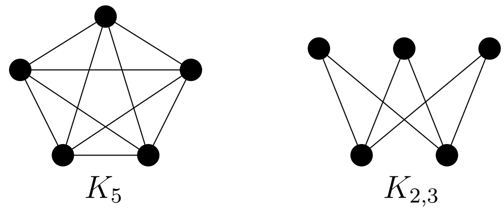

Figure 1.2.3: On the left hand side it is illustrated a complete graph on 5 vertices and on the right hand side a complete bipartite graph.

Definition 1.2.33 (k-partite graph, bipartite graph)**.** Let $k \geq 2$ be a positive integer. A graph G will be said to be a *k-partite graph* if and only if its vertex set $V(G)$ admits a partition into k sets such for every edge e of G the endpoints e belong to different sets of the partition, i.e. for each set of the partition no two vertices of this set are adjacent. A 2-partite graph is called *bipartite*.

Definition 1.2.34 (complete k-partite graph, complete bipartite graph)**.** Let G be k-partite graph and consider a partition of $V(G)$ which witness our assumption. The graph G will be said to be *complete k-partite graph* if and only if every two vertices which belong to different sets of the partition are adjacent. A complete 2-partite graph is called complete bipartite graph.

1.2.1 Operations on graphs & graphs' relations

Definition 1.2.35 (vertex deletion)**.** Let G be a graph. The *deletion* of a vertex $v \in V(G)$ transforms G to the graph $(V(G) \setminus \{v\}, \{e \in E(G) | v \notin e\})$. The resulting graph after applying the deletion of a vertex v on G is denoted by $G \setminus v$.

Definition 1.2.36 (edge deletion)**.** Let G be a graph. The *deletion* of an edge $e \in E(G)$ transforms G to the graph $(V(G), E(G) \setminus \{e\})$. The resulting graph after applying the deletion of an edge e on G is denoted by $G \setminus e$.

Definition 1.2.37 (edge contraction, contraction vertex)**.** Let G be a graph and let $e = \{u, v\}$ be an edge of G. The *contaction* of the edge e is the operation which consists in the deletion of e and the addendum of a new vertex v_e to the graph G which we connect with all the neighbours of u and v (if some multiple edges are created we delete them). Thus the contraction of an edge e on a graph G yields to the graph:

$$\big(V(G) \cup \{v_e\}, \{E(G) \setminus e\} \cup \{\{v_e, v'\} | (v' \in N_G(u)) \vee (v' \in N_G(v))\}\big),$$

which is denoted by G/e. The vertex v_e will be said to be the *contraction vertex*.

Definition 1.2.38 (suppression of a vertex)**.** Let G be a graph and v be a vertex of degree 2 of G, then the *suppression* of v consists in the contraction of the one of the two edges which are incident to v.

Definition 1.2.39 (subdivision of an edge, subdivision vertex)**.** Let G be a graph and let $e = \{u, v\}$ be an edge of G. The *subdivision* of G is the operation which consists in the deletion of e and the addendum of a new vertex v_e, and the two edges $\{u, v_e\}, \{v_e, v\}$ in G. Thus the contraction of an edge e on a graph G yields to the graph:

$$\big(V(G) \cup \{v_e\}, \{E(G) \setminus e\} \cup \{\{u, v_e\}, \{v_e, v\}\}\big).$$

Definition 1.2.40 (lift of two edges)**.** Let G be a graph and let $e_1 = \{x, y\}$ and $e_2 = \{x, z\}$ be two edges of G. The *lift* of those two edges is the operation which consists in the removal of e_1 and e_2 from G and the addendum of the edge $\{y, z\}$. Thus the lift of two edges $e_1 = \{x, y\}$ and $e_2 = \{x, z\}$ on a graph G yields to the graph:

$$\big(V(G), E(G) \setminus \{e_1, e_2\} \cup \{\{y, z\}\}\big).$$

Comment 1.2.41. *Notice that in the case the edge we are adding already present, the lift of two edges may create a multiple edge. this is the only exception throughout this thesis where multiple edges -as a result of the application of an operation on graphs- are allowed.*

Based on the above operations we define several binary relations between graphs.

Graph 's relation	vertex deletion	vertex dissolution	edge deletion	edge contraction	lift
Subgraph	•		•		
Spanning subgraph			•		
Induced subgraph	•				
Minor	•		•	•	
Topological minor	•	•	•		
Induced minor	•			•	
Weak immersion	•		•		•

Table 1.2: Graphs' relations and correspondence operations.

Definition 1.2.42 (subgraph relation on graphs)**.** Given two graphs, say H and G, the graph H will be said to be a *subgraph* of G -denoted by $H \leq G$- if and only if there exist a natural number n and a sequence of graphs $G_0, \ldots, G_n$ such that: $G = G_0$, $H = G_n$ and for each $i \in \{1, \ldots, n\}$, the graph G_i can be obtained from the graph G_{i-1} by the deletion of one vertex or the deletion of one edge.

Definition 1.2.43 (induced subgraph relation on graphs)**.** Given two graphs, say H and G, the graph H will be said to be an *induced subgraph* of G -denoted by $H \leq_{is} G$- if and only if there exist a natural number n and a sequence of graphs $G_0, \ldots, G_n$ such that: $G = G_0$, $H = G_n$ and for each $i \in \{1, \ldots, n\}$, the graph G_i can be obtained from the graph G_{i-1} by the deletion of a vertex.

Definition 1.2.44 (topological minor relation on graphs)**.** Given two graphs, say H and G, the graph H will be said to be a *topological minor* of G -denoted by $H \leq_{tm} G$- if and only if there exist a natural number n and a sequence of graphs $G_0, \ldots, G_n$ such that: $G = G_0$, $H = G_n$ and for each $i \in \{1, \ldots, n\}$, the graph G_i can be obtained from the graph G_{i-1} by the deletion of a vertex, or the deletion of an edge, or by the suppression of a vertex

Definition 1.2.45 (induced minor relation on graphs)**.** Given two graphs, say H and G, the graph H will be said to be a *induced minor* of G -denoted by $H \leq_{im} G$- if and only if there exist a natural number n and a sequence of graphs $G_0, \ldots, G_n$ such that: $G = G_0$, $H = G_n$ and for each $i \in \{1, \ldots, n\}$, the graph G_i can be obtained from the graph G_{i-1} by the deletion of a vertex or by the contraction of an edge.

Definition 1.2.46 (contraction relation on graphs)**.** Given two graphs, say H and G, the graph H will be said to be a *contraction* of G -denoted by $H \leq_c G$- if and only if there exist a natural number n and a sequence of graphs $G_0, \ldots, G_n$ such that: $G = G_0$, $H = G_n$ and for each $i \in \{1, \ldots, n\}$, the graph G_i can be obtained from the graph G_{i-1} by the contraction of an edge.

Comment 1.2.47. *Presented two graphs H, G such that H is a contraction of G we will also say that the graph G can be contracted onto H.*

Definition 1.2.48 (subdivision relation on graphs)**.** Given two graphs, say H and G, the graph H will be said to be a *subdivision* of G -denoted by $H \leq_s G$- if and only if there exist a natural number n and a sequence of graphs $G_0, \ldots, G_n$ such that: $G = G_0$, $H = G_n$ and for each $i \in \{1, \ldots, n\}$, the graph G_i can be obtained from the graph G_{i-1} by the subdivision of an edge.

Definition 1.2.49 (weak immersion relation on graphs)**.** Given two graphs, say H and G, the graph H will be said to be a *weak immersion* of G -denoted by $H \leq_{im}^{w} G$- if and only if there exist a natural number n and a sequence of graphs $G_0, \ldots, G_n$ such that: $G = G_0$, $H = G_n$ and for each $i \in \{1, \ldots, n\}$, the graph G_i can be obtained from the graph G_{i-1} by the deletion of a vertex, or by the deletion of an edge, or by a vertex suppressions.

Definition 1.2.50 (strong immersion relation on graphs)**.** Given two graphs, say H and G, we will say that the graph H can be *strongly immersed in* the graph G, if and only if H can be obtained from G by a sequence of vertex splittings (i.e., lifting some pairs of incident edges and removing the vertex) and edge removals.

Definition 1.2.51 (*S-maintaining contraction*)**.** Let G be a graph, let $S \subseteq V(G)$, and let $e = \{x, y\}$ be an edge of $E(G)$ such that not both its endpoints are in S. We say that G' is the result of a *S-maintaining contraction* in G if G' is obtained if we remove x and y from G, add a new vertex v_{new} and make it adjacent with all vertices in the neighborhood of x or y in G that are still vertices in G'. In the resulting graph, in case one, say x, of x and y is a member of S, we rename v_{new} by x.

Notation 1.2.52. Let X and Y be sets. Let also $\alpha : X \to Y$ and $\sigma : X \to 2^Y$. We will write $\alpha \Subset \sigma$ if and only if $(\forall x \in X)[\alpha(x) \subseteq \sigma(x)]$.

Definition 1.2.53 (*α-rooted minor*)**.** Let G and H be graphs and let $\alpha : V(H) \to V(G)$ be some function mapping vertices of H to vertices of G . We say that H is an *α-rooted minor* of G if there is a function $\sigma : V(H) \to 2^{V(G)}$ where $\alpha \Subset \sigma$ and such that

- $\forall x \in V(H)$ $G[\sigma(x)]$ is a connected graph;
- $\forall x, y \in V(H)$ $x \neq y \Rightarrow \sigma(x) \cap \sigma(y) = \emptyset$; and
- $\forall \{x, y\} \in E(H)$ $G[\sigma(x) \cup \sigma(y)]$ is a connected graph.

We call the function σ *minor model of* H *in* G.

The next observation offers a more dynamic way to define the notion of a α-rooted minor.

Observation 1.2.54. H is an α-rooted minor of G if and only if there is a sequence of $\alpha(V(H))$-maintaining contractions, edge removals, or vertex removals, that transform G to a graph H^* such that H is isomorphic to H^* via some isomorphism $\rho : V(H) \to V(H^*)$ where $\alpha \subseteq \rho$.

Definition 1.2.55 (minor)**.** We say that H is a *minor* of G -denoted by $H \leq_m G$- if and only if H is a $\varnothing$-minor of G.

The following is an immediate corollary of the above definition.

Corollary 1.2.56. Let H, G be two graphs, then H is a minor of G if and only if there exist a natural number n and a sequence of graphs $G_0, \ldots, G_n$ such that: $G = G_0$, $H = G_n$ and for each $i \in \{1, \ldots, n\}$, the graph G_i can be obtained from the graph G_{i-1} by the deletion of a vertex, or by the deletion of an edge, or by a contraction of an edge.

Observation 1.2.57. There exists no infinite strictly decreasing sequence of finite graphs with respect to any of the aforementioned graphs' relations.

Notation 1.2.58. Throughout this thesis when we say that a graph *has an* H*-minor* we mean that it has a minor isomorphic to H. We use analogue phrases for all the other graphs' relations that we have denote in this subsection, for example sometimes we say that G has an H-subgraph instead of saying G has a subgraph isomorphic to H.

Finally we have the following relation between minors and topological minors:

Proposition 1.2.59. Let G be a graph, then

(i) Every topological minor of G is also a minor of G;

(ii) If $\Delta(G) \leq 3$, then every minor of G is also a topological minor of G.

Definition 1.2.60 (graph property closed under a relation)**.** Let $\mathcal{Q}$ be a graph property and $\leq$ be a binary relation on graphs. The property $\mathcal{Q}$ will be said to be *closed under the relation* $\leq$ if and only if ($\forall$ graphs G, H)$[(G \leq H) \wedge (H$ satisfy $\mathcal{Q}) \Rightarrow G$ satisfy $\mathcal{Q}]$

1.2.2 Connectivity

Definition 1.2.61 (connected graph)**.** A graph G will be said to be *connected* if and only if $G \neq (\emptyset, \emptyset)$ and any two of its vertices are linked by a path in G.

Definition 1.2.62 (acyclic)**.** A graph will be said to be *acyclic* if and only if it has not a cycle as a subgraph.

Definition 1.2.63 (forest, tree, leaf of a tree, inner vertices of at tree)**.** A graph will be said to be a *forest* if and only if it is acyclic. A connected forest will be said to be a *tree*. A vertex of tree will be called a *leaf* if and only if has degree 1. Any vertex of tree which is not a leaf will be said to be an *inner* vertex of the tree.

Notation 1.2.64 (set of leaves of a tree)**.** Let T be a tree. We denote by $L(T)$ the set of its leaves.

Definition 1.2.65 (cubic tree)**.** A tree will be said to be *cubic* if and only if every inner vertex of it has degree 3.

Definition 1.2.66 (connected component)**.** Let G be a graph a subgraph H of G will be said to be a *connected component* of G if and only if H is maximal connected -with respect to the subgraph- among the subgraphs of G, that is

$$\big(H \text{ is connected}\big) \wedge \big((\forall F)[(F \subseteq G) \wedge (H \subseteq F) \rightarrow F \text{ is not connected}]\big).$$

Definition 1.2.67 ((A, B)-separator, minimal/minimum (A, B)-separator)**.** Let G be a graph and let $A, B, S \subseteq V(G)$. The set S will be said to be an (A, B)*-separator* in G if and only if S contains at least one vertex of every (A, B)-path of G. Given two vertices, say a, b of G the set of vertices S will be said to be an (a, b)*-separator* if and only if S it is an $(\{a\}, \{b\})$-separator. In the case that S is an (a, b)-separator we say that S separates a from b in G. An (A, B)-separator is said to be a *minimal* (A, B)*-separator* if and only if every proper subset of S is not an (A, B)-separator. An (A, B)-separator S is said to be a *minimum* (A, B)*-separator* if and only if every other (A, B)-separator in G contains more vertices than S.

Definition 1.2.68 (separator, minimal separator, minimum separator)**.** Let G be a connected graph and $S \subseteq V(G)$. The set of vertices S will be said to be a *separator* of G if and only if it separates any two vertices of G. A separator S is said to be a *minimal separator* if and only if every proper subset of S is not a separator. A separator S is said to be a *minimum separator* if and only if every other separator of G contains more vertices than S.

Definition 1.2.69 (bridge)**.** Let G be a graph and e be an edge of G. The edge e will be said to be a *bridge* of G if and only if e do not lie in any cycle of G.

Definition 1.2.70 (cut-vertex)**.** Let G be a graph. A vertex v of G will be said to be a *cut-vertex* if and only if the set $\{v\}$ is separator of a connected component of G.

Definition 1.2.71 (k-connected graph)**.** Let k be a positive integer, a graph G will be said to be k*-connected* if and only if it $|V(G)| > k$ and every separator of G has at least k vertices.

Definition 1.2.72 (block of a graph)**.** Let G be a graph. A subgraph H of G will be said to be a *block* of G if and only if it is maximally connected and it none of its vertices is a cut-vertex of H. Thus, every block of G is either maximal 2-connected subgraph of G, or a bridge, or an isolated vertex.

The following theorem of Karl Menger is one of the cornerstones of graph theory, and we will use it extensively in further chapters.

Theorem 1.2.73 (Menger [91])**.** For every graph G and given any two sets , say A, B, of vertices of G, the cardinality of the minimum (A, B)-separator equals to the maximum number of pairwise internally vertex-disjoint (A, B)-paths.

Having formulate the basic definitions of graphs, we can now state, for historical interest, the *Seven Bridges of Königsberg Problem* which typically consists the "birth certificate" of graph theory as a branch of mathematics.

First is needed to define what an Eulerian cycle is.

Definition 1.2.74 (Eulerian cycle, Eulerian graph)**.** A closed walk in a graph will be said to be an *Eulerian cycle* if and only if it traverse every edge of the graph exactly once. A graph will be said to be *Eulerian* if and only if it admits[4] an Eulerian cycle.

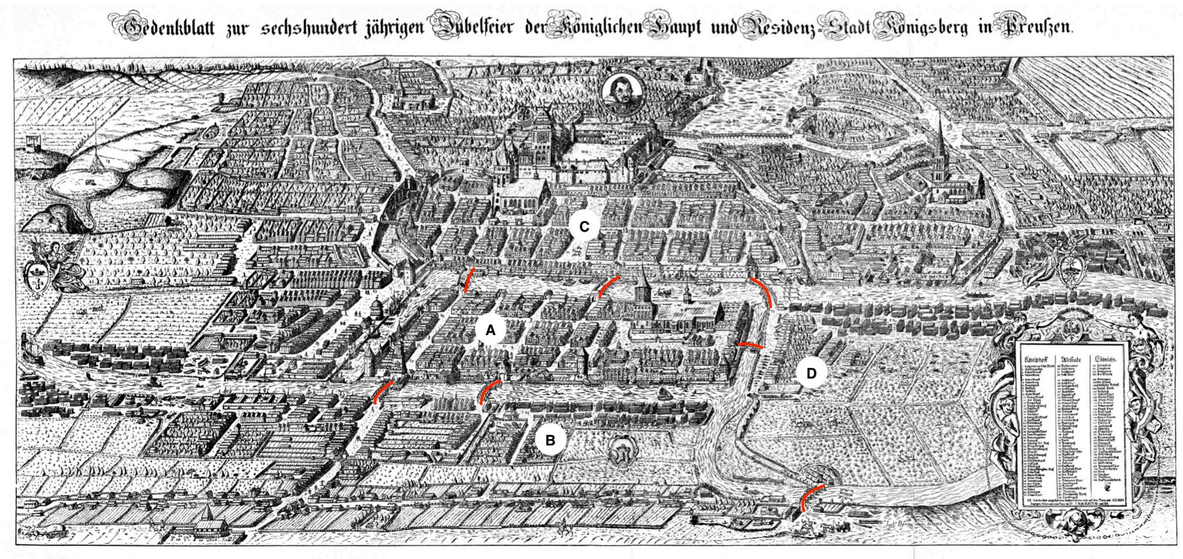

Figure 1.2.4: The seven bridges of Königsberg.

«It said that the people of Königsberg[5] used to entertain themselves by trying to devise a route around the city which would cross each of the seven bridges just once. Since their attempts had always failed, many of them believed that the task was impossible, but it was not until the 1730s that the problem was treated from a mathematical point of view and the impossibility of finding such a root was proved. In 1736, one of the leading mathematicians of the time, Leonhard Euler, communicate with other mathematicians in the problem, and gave a general method for other problems of the same type»[6] So the the Königsberg bridge problem can be formulated as follows:

> Can all the seven bridges of the city of Königsberg (Figure 1.2.4), be traversed in a single trip without doubling back, with the additional requirement that the trip ends in the same place it began?

Euler's treatment of the Königsberg bridge problem involved two major steps. First he replaced the map of the city by a simple diagram (that is, by a graph!) which was encapsulate only those informations which were necessary for the problem and then he formulate the problem in such a way that the diagram became unnecessary.[7] In nowadays -graph theory- terms the Königsberg bridge problem can be formulated as follows:

[4] We shall often use some terms which we have not formally define, as long as their meaning is obvious.

[5] A/N. On 4 July 1946 the Soviet authorities renamed Königsberg to Kaliningrad.

[6] Biggs, Lloyd, and Wilson [13, page 2]

[7] For an analytical presentation of the article in which Euler dealt with this problem and in general of the history of graph theory, we refer the interested reader in [13].

Does the underlying graph (Figure 1.2.5), where bridges correspond to edges, admits an Eulerian cycle.

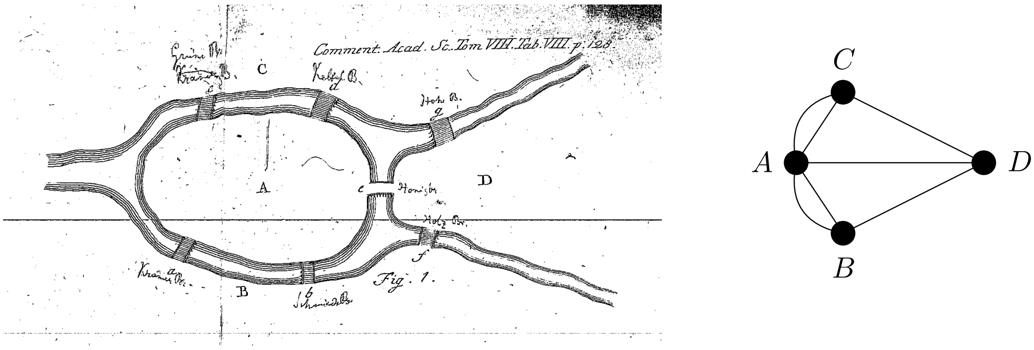

Figure 1.2.5: On the left hand side a figure from Euler's paper [46] and on the right hand side is illustrated the multigraph which correspond to the seven bridges of Königsberg problem.

Euler [46] proved in 1736 the following theorem.

Theorem 1.2.75 (Euler [46], 1736)**.** A connected graph is Eulerian if and only if every vertex of the graph has even degree.

1.2.3 Directed graphs

Definition 1.2.76 (directed graph)**.** A *directed graph* is an ordered pair of two sets such that the second set of this pair is subset of the Cartesian square of the first set. The elements of the first set of the ordered pair are called *vertices* of the graph and those of the second set are called *edges* of the graph. That way the first set of the ordered pair will be called the *vertex set* or the set of vertices of the graph and the second will be called *edge set* of the graph.

Notation 1.2.77. Let D be a directed graph. The vertex set and the edge set of D will be denoted by $V(D)$ and $E(D)$ respectively. This notation is independent of the specific name of these sets and is depended only on the name of the graph.

Definition 1.2.78 (head and tail of an edge of a directed graph, loop edge)**.** Let D be a directed graph and $e = (u, v) \in E(D)$ be an edge of D, then the vertex v will be called the *head* of the edge e and will be denoted by head(e) and the vertex u will be called the *tail* of e and will be denoted by tail(e). The edge e will be said to be an edge from u to v. If $u = v$ then the edge e will be said to be a *loop*.

Definition 1.2.79 (multiple and parallel edges of a directed graph)**.** Note that a directed graph may have several edges between the same two vertices, say u, v. Such edges are called *multiple* edges; if they have the same direction (say from u to v), they are *parallel*.

Definition 1.2.80 (orientation of an undirected graph, underlying graph of a directed graph)**.** A directed graph D is an *orientation* of an undirected graph G if and only if

$$\big((V(D) = V(G)) \land (E(D) = E(G))\big) \land \big((\forall e \in E(G)[e = \{x, y\} \to \{head(e), tail(e)\} = \{x, y\}])\big).$$

Given a directed graph D if there exists a graph G such that D is an orientation of G then then the graph G will be said to be the *underlying* (undirected) graph of D.

Comment 1.2.81. *Throughout this thesis, if not stated otherwise, the graphs that we consider will be finite and undirected.*

1.3 Topology

In this section, we present those definitions from topology that are necessary, in order to formally define in the next section surfaces. For a rigorous introduction in topology of surfaces, we refer the interested reader to [74].

Definition 1.3.1 (topology on a set)**.** Let X be a set. If $\mathfrak{T} \subseteq \mathcal{P}(X)$, then $\mathfrak{T}$ will be said to be a *topology* on X, if and only if:

(i) $\emptyset, X \in \mathfrak{T}$;

(ii) $\mathfrak{T}$ is closed under finite intersections, i.e. if $n \in \mathbb{N}$ and $G_1, \ldots, G_n \in \mathfrak{T}$, then $\bigcap_{i=1}^{n} G_i \in \mathfrak{T}$;

(iii) $\mathfrak{T}$ is closed under unions, i.e. if I is a set and $(\forall i \in I)[G_i \in \mathfrak{T}]$, then $\bigcup_{i \in I} G_i \in \mathfrak{T}$.

Definition 1.3.2 (topological space)**.** A *topological space* is an ordered pair $(X, \mathfrak{T})$, such that X is a set and $\mathfrak{T}$ is a topology on X.

Definition 1.3.3 (point of topological space)**.** Let $(X, \mathfrak{T})$ be a topological space, any element of X we be called also a *point* of X, or a *point of the topological space* when it is clear from the context to which topological space we refer.

Comment 1.3.4. *Given a set X on which is defined a topology, when it is clear from the context which is that topology or when it does not matter for our purposes which is the specific topology we may identify our reference to the topological space $(X, \mathfrak{T})$ (which $\mathfrak{T}$ is the topology defined on X) with our reference to the set X, that is, we call the set X a topological space.*

Definition 1.3.5 (open sets)**.** Let $(X, \mathfrak{T})$ be a topological space. The elements of $\mathfrak{T}$ are called *open sets* (of $\mathfrak{T}$ or of $(X, \mathfrak{T})$).

Definition 1.3.6 (closed sets)**.** Let $(X, \mathfrak{T})$ be a topological space and let F be a subset of X. The set F will be said to be *closed* if and only if its complement is an open set, that is F is closed if and only if $X \setminus F \in \mathfrak{T}$.

The proposition below follows immediate form the definitions of closed and open sets and De Morgan's laws.

Proposition 1.3.7. Let $(X, \mathfrak{T})$ be a topological space. Then the following hold:

(i) $\emptyset$ and X are closed sets;

(ii) the union of finitely many closed sets is a closed set;

(iii) the intersection of arbitrary many closed sets is a closed set.

Definition 1.3.8 (neighbourhood of a point)**.** Let $(X, \mathfrak{T})$ be a topological space and x be a point of it. A *neighbourhood* of x is any set V which has an open subset $G \subseteq V$ such that $x \in G$.

Definition 1.3.9 (subspace\relative topology, subspace of a topological space)**.** Let $(X, \mathfrak{T})$ be a topological space and let also $Y \subseteq X$, then the set $\mathfrak{T}_Y$ is a topology on Y. The topology $\mathfrak{T}_Y$ is called the *subspace topology* or the *relative topology* w.r.t $\mathfrak{T}$ and the topological space $(Y, \mathfrak{T}_Y)$ is called a *subspace* of the topological space X.

Definition 1.3.10 (Hausdorff topological space)**.** A topological space $(X, \mathfrak{T})$ will be called *Hausdorff* if and only if:

$$(\forall x, y \in X)(\exists G_1, G_2 \in \mathfrak{T})[(x \in G_1) \wedge (y \in G_2) \wedge (G_1 \cap G_2 = \emptyset)]$$

Definition 1.3.11 (connected topological space)**.** A topological space $(X, \mathfrak{T})$ will be said to be *connected* if and only if X cannot be written as a union of two non-empty disjoint open sets.

Definition 1.3.12 (continuous function between two topological spaces)**.** Let $(X, \mathfrak{T}_X)$ and $(Y, \mathfrak{T}_Y)$ be two topological spaces. Then given a function $f : X \longleftrightarrow Y$, f will be said to be *continuous* if and only if whenever a set A is an open set in Y, $f^{-1}(A)$ is an open set in X, that is, $(\forall A \in \mathfrak{T}_Y)[f^{-1}(A) \in \mathfrak{T}_X]$.

Definition 1.3.13 (homeomorphism between topological spaces)**.** Let $(X, \mathfrak{T}_X)$ and $(Y, \mathfrak{T}_Y)$ be two topological spaces. Then given a function $f : X \longleftrightarrow Y$ will be said to be an *homeomorphism* from X to Y if and only if f is one-to-one, onto, continuous and the function $f^{-1} : Y \to X$ is also continuous

Definition 1.3.14 (homeomorphic topological spaces)**.** Two topological spaces, say $(X, \mathfrak{T}_X)$ and $(Y, \mathfrak{T}_Y)$ will be said to be *homeomorhic* -denoted by $X \simeq Y$- if and only if there exist a homeomorphism $f : X \to Y$.

Definition 1.3.15 (open cover, finite cover, open subcover)**.** Let $(X, \mathfrak{T}_X)$ be a topological space and $A \subseteq X$. A *open cover* of A is any collection $\mathcal{U}$ of open sets of the topological space X such that the set A is a subset of their union, that is, $\big(\mathcal{U} \subseteq \mathfrak{T}_X\big) \wedge \big(A \subseteq \bigcup_{u \in \mathcal{U}} U\big)$. Given an open cover $\mathcal{U}$ of a set A a subcollection $\mathcal{U}' \subseteq \mathcal{U}$ will be said to be a *subcover* if and only if $\mathcal{U}'$ is an open cover of A. A cover is said to be a *finite cover* if and only if it consists in finitely many sets.

Definition 1.3.16 (compact topological space)**.** A topological space X is called *compact* if and only if every open cover of X has a finite subcover.

Definition 1.3.17 (Euclidean metric\ordinary metric, Euclidean distance of two points)**.** Let n be a positive natural number, then the *Euclidean metric* or the *ordinary metric* on the set $\mathbb{R}^n$, is the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is such that:

$$(\forall x, y \in \mathbb{R}^n)[\big(x = (x_1, \ldots, x_n)\big) \wedge \big(y = (y_1 \ldots y_n)\big) \to d(x, y) = \sqrt{(x_1 - y_1)^2 \ldots (x_n - y_n)^2}].$$

Given two points, say x, y, of $\mathbb{R}^n$, the *Euclidean distance* of those points -denoted by $\|x - y\|$- is the non-negative real number $d(x, y)$

Definition 1.3.18 (Euclidean norm\Euclidean length)**.** Let n be a positive natural number and $x \in \mathbb{R}^n$ the non-negative real number $d(x, 0)$ which is laso denoted by $\|x\|$ will be said to be the *Euclidean norm* or the *Euclidean length* of x.

Definition 1.3.19 (open disc, closed disc, sphere of $\mathbb{R}^{\ltimes}$)**.** Let n be a positive natural number, r be a positive real number and $x \in \mathbb{R}^n$. Then

- The set $\{y \in \mathbb{R}^n | \|x-y\| < r\}$ -denoted by $D^n(x,r)$- will be said to be the *open disc* with center x and radius r;
- the set $\{y \in \mathbb{R}^n | \|x-y\| \leq r\}$ -denoted by $\bar{D}^n(x,r)$- will be said to be the *closed disc* with center x and radius r;
- the set $\{y \in \mathbb{R}^n | \|x-y\| = r\}$ -denoted by $S^n(x,r)$- will be said to be the *sphere* with center x and radius r.

Definition 1.3.20 (Euclidean topology on $\mathbb{R}^n$)**.** Let n be a positive natural number, then it is easy to see that the set

$$\{G \subseteq \mathbb{R}^n | (\forall x \in G)(\exists r \in \mathbb{R}^+)[\big(D(x,r) \subseteq G\big) \wedge \big(x \in D(x,r)\big)]\}$$

is a topology on $\mathbb{R}^n$. This topology is called the *Euclidean topology on* $\mathbb{R}^n$ and the set $\mathbb{R}^n$ supplied with this topology is the *Euclidean space*.

Comment 1.3.21. *When we refer to* $\mathbb{R}^n$ *as a topological space, we mean the Euclidean space.*

1.3.1 Creating new topological spaces from old ones

In topology we often want to construct more complex objects from simpler ones using «gluing» methods. These situations may, at first glance, look different, but they are essentially manifestations of a general construction. The concept of quotation topology essentially encompasses the typical description of this construction.

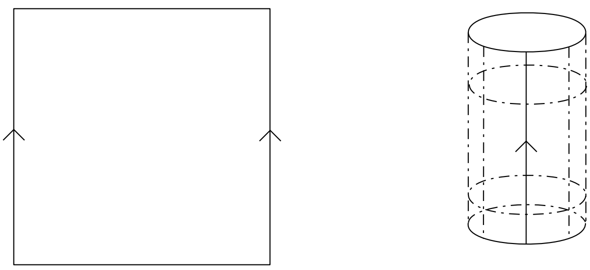

Figure 1.3.1: Identifying the opposite sides of a rectangle in order to create a cylinder.

Definition 1.3.22 (quotient topology)**.** Let $(X, \mathfrak{T})$ be a topological space, let Y be a set and $f : X \to Y$ be an surjection. We definite the quotient topology $\mathfrak{T}'$ on Y as follows: $\mathfrak{T}' := \{U \subseteq Y | f^{-1}(U) \in \mathfrak{T}\}$.

Comment 1.3.23. *Note that in the above -provided the topological space* $(X, \mathfrak{T})$*, the set* Y *and the function* f*- the topology* $\mathfrak{T}$ *is the smallest one (w.r.t. inclusion), with which we can supply the set* Y *in order to make the function* f *continuous.*

Definition 1.3.24 (identification topology)**.** Let $(X, \mathfrak{T})$ be a topological space and let also $\sim$ be an equivalent relation on X then the *identification space* of X w.r.t $\sim$ -denoted by $X/\sim$ - is defined to be the set of the equivalent classes of the relation $\sim$, that is, $X/\sim := \{[x]^{\sim} \ |x \in X\}$.

Comment 1.3.25. *The new space $X/\sim$ is just a fancy way of saying that a new space is created by taking the space X and gluing x to any y that satisfies $y \sim x$.*

1.4 Surfaces

In this section we introduce those elements of surfaces which are needed for the understanding of the proof of the "Kuratowski theorem for general surfaces" (Theorem 3.1.5). For this section we followed Appendix B of [30].

Definition 1.4.1 (surface)**.** A *surface* is a compact connected Hausdorff topological space Σ in which every point has a neighbourhood which is homeomorphic (as a topological space with the subspace topology) to the Euclidean plane $\mathbb{R}^2$.

Comment 1.4.2 (closed surfaces)**.** *A surface is closed if it is compact, connected and has no boundary, here we consider closed surfaces.*

Definition 1.4.3 (unit circle)**.** The set $S^2(0,1) \subseteq \mathbb{R}^2$ will be called the *unit circle* and will be denoted by S^1.

Definition 1.4.4 (sphere)**.** The set $S^3(0,1) \subseteq \mathbb{R}^2$ will be called the *sphere* and will be denoted by S^2.

Definition 1.4.5 (cylinder)**.** The *cylinder* is the topological space contracted from the subspace *unit square* X of $\mathbb{R}^2$, where $X = \{(x,y)|(0 \leq x \leq 1) \wedge (0 \leq y \leq 1)\}$ by identifying its opposite sides. The *middle cycle* of the cylinder is the subspace $D^1(0,1) \times \{\frac{1}{2}\}$.

Definition 1.4.6 (arc\cycle\open disc\closed disc\disc in a surface)**.** Given a surface Σ, an *arc*, a *circle*, an *open disc* and a *closed disc* in Σ, are subsets of Σ which are homeomorphic (as topological spaces equipped with the subspace topology) to the real interval $[0,1]$, to the unit cycle S^1, to the open disc $D^2(0,1)$ with center 0 and radius 1 and to the closed disc $\bar{D}^2(0,1)$ with center 0 and radius 1 respectively. A *disc* in Σ is any open or closed disc in Σ.

Definition 1.4.7 (components of a surface)**.** Let Σ be a surface and $X \subseteq \Sigma$ and consider the following binary relation on X:

$$\sim := \{(x,y)|\big((x,y) \in X \times X\big) \wedge \big(\text{there is an arc in } \Sigma \text{ from } x \text{ to } y\big)\}.$$

It is easy to see that the relation $\sim$ is reflexive, symmetric and transitive and thus $\sim$ is an equivalent relation on X. The equivalence classes of points in X with respect to $\sim$ will be said to be the *components* of X. Thus, two points of a set in a surface belong to the same component of the set if and only if they can be joined by an arc.

Observation 1.4.8. Every surface being by definition connected has only one component.

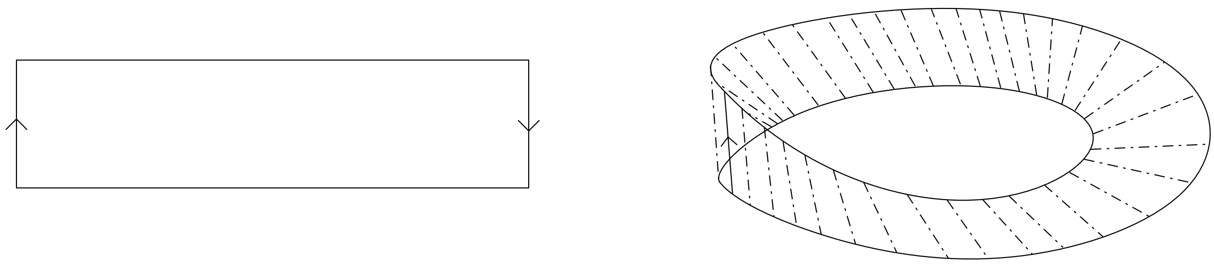

Figure 1.4.1: Deducing the Möbius strip by identifying the opposite sides of a rectangle in the appropriate manner.

Definition 1.4.9 (frontier of a set, boundary cycle)**.** Given a surface Σ and a subset X of Σ, the set

$$\{y \in \Sigma | (\forall U \subseteq \Sigma)[U \text{ is a neighbourhood of } y \to (U \cap X \neq \emptyset) \wedge (U \cap (\Sigma \setminus X) \neq \emptyset)\}]$$

will be said to be the *frontier* of X. The frontier F of X *separates* $\Sigma \setminus X$ from X: since $X \cup F$ is closed, every arc from $\Sigma \setminus X$ to X has a first point in $X \cup F$, which must lie in F. A component of the frontier of X that is a cycle in Σ is a *boundary cycle* in X. A boundary cycle of a disc in Σ will be said to *bound* that disc.

Definition 1.4.10 (Möbius strip, middle cycle of Möbius strip)**.** A *Möbius strip* is any space homeomorphic with the topological space $[0, 1] \times [0, 1]$ with the Euclidean Topology, after the identification of any two points $(1, y), (0, 1 - y)$ for all $y \in [0, 1]$. Its *middle cycle* is the subspace $\{(x, \frac{1}{2}) | 0 < x < 1\} \cup \{p\}$, where p is the point resulting from the identification of the $(1, \frac{1}{2})$ with $(0, \frac{1}{2})$.

Definition 1.4.11 (strip neighborhood, two-sided, one-sided)**.** It can be shown that any cycle C in a surface Σ is the middle cycle of a suitable cylinder or Möbius strip N. This cylinder or Möbius strip is called *strip neighborhood* of C. If this strip neighborhood is a cylinder, then then $N \setminus C$ has to components and we call the cycle C *two sided*; if N is a Möbius strip, then $N \setminus C$ has only one component and C is called *one-sided*.

Definition 1.4.12 (separating cycle)**.** Since any surface Σ is connected, given a cycle Σ, $\Sigma \setminus C$ cannot have more components than $N \setminus C$, (where N is the strip neighborhood of C), we call C separating cycle if $\Sigma \setminus C$ has two components and *non-separating* if$\Sigma \setminus C$ has only one component.

Below we describe the two main operations, which we can apply in a simper surface and take a more complex one. Actually *the classification theorem* (Theorem 1.4.15) states that by applying finitely many times these operations in a sphere we can take any surface.

Definition 1.4.13 (adding a handle to a surface)**.** To *add a handle* to a surface Σ, we remove two open discs whose closures in Σ are disjoint, and identify with the cycles $S^1 \times \{0\}$ and $S^1 \times \{1\}$ of a copy of the cylinder $S^1 \times [0, 1]$ disjoint from Σ.

Definition 1.4.14 (adding a crosscap)**.** To *add a crosscap* in a surface Σ, we remove an open disc from Σ and we identify opposite points on its boundary cycle in pairs.

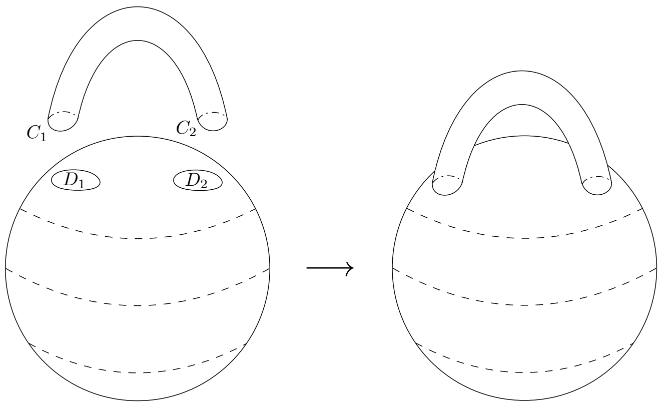

Figure 1.4.2: Illustration of Definition 1.4.13. In the case illustrated above we are adding a handle to a sphere, D_1, D_2 are two open disks whose closures are disjoint in Σ, the operation of *adding a handle* consists in identifying the boundaries of these open disks with the cycles C_1, C_2 respectively.

Theorem 1.4.15 (The classification theorem)**.** Any closed surface is homeomorphic either to the sphere, or to the sphere with a finite number of handles added, or to the sphere with a finite number of discs removed and replaced by Möbius strips. No two of the aforementioned surfaces are homeomorphic.

Definition 1.4.16 (Embedding of a graph in a surface, face of a graph in a surface, boundary of a face)**.** Let G be a graph and Σ be a surface. An *embedding* of G in Σ is a map σ with domain the set $V(G) \cup E(G)$, that maps the vertices of G to distinct points in Σ and its edges $\{x, y\}$ to $\sigma(x) - \sigma(y)$ arcs in Σ, so that no inner point of such an arc is the image via σ of a vertex or lies in another arc. We then write $\sigma(G)$ for the union of all those points and arcs in Σ i.e., for $\sigma(V(G)) \cup \sigma(E(G))$. A *face* of G in Σ is a component of $\Sigma \setminus \sigma(G)$, and the subgraph of G that σ maps to the frontier of this face is its *boundary*. Note that while faces in the sphere are always discs (if G is connected), in general they need not be.

Definition 1.4.17 (planar graph)**.** A graph will be said to be *planar* if and only if it can be embedded in the sphere.

Definition 1.4.18 (plane graph)**.** A *plane* graph is an ordered pair (V, E) of finites sets with the following properties (the elements of V are called again vertices and the elements of E edges):

(i) $V \subseteq \mathbb{R}^2$;

(ii) every edge is an arc between two vertices;

(iii) different edges have different sets of endpoints;

(iv) the interior of an edge contains no vertex and no point of any other edge.

A plane graph (V, E) defines a graph G on V in a natural way. As long as no confusion can arise, we can use the name G of this abstract graph also for the plane graph (V, E) or for the point set

$V \cup \left(\bigcup_{e \in E} e\right)$. Similar notational conventions will be used for abstract versus plane edges, for subgraphs, and so on.

We state some theorems and lemmas below without proofs.

Lemma 1.4.19. Any two planar embeddings of a 3-connected graph are equivalent.

Lemma 1.4.20. Every surface other than the sphere contains a non-separating cycle.

Theorem 1.4.21. For every surface Σ there exist an integer $\chi(\Sigma)$ such that whenever a graph G eith n vertices and m edges is embedded in Σ so that there are l faces and every face is a disc, we have

$$n - m + l = \chi(\Sigma).$$

Definition 1.4.22 (Euler characteristic of a surface)**.** For every surface Σ the integer $\chi(\Sigma)$ of the above theorem is said to be the *Euler characteristic* of Σ.

Definition 1.4.23 (Euler genus of a surface)**.** For every surface Σ the natural number $2 - \chi(\Sigma)$ is said to be the *Euler genus* of Σ and it is denoted by $\varepsilon(\Sigma)$.

Lemma 1.4.24.

(i) Adding a handle to a surface raises its Euler genus by 2.

(ii) Adding a crosscap to a surface raises its Euler genus by 1.

The following two lemmas will be used at the proof of Theorem 3.1.5.

Lemma 1.4.25 ([30, Lemma B6])**.** Let Σ be a surface, and let $\mathcal{C}$ be a finite set of disjoint cycles in Σ. Assume that $\Sigma \setminus \bigcup \mathcal{C}$ has a component D_0 whose closure in Σ meets every cycle in $\mathcal{C}$, and that no cycle in $\mathcal{C}$ bounds a disc in Σ that is disjoint from D_0. Then $\varepsilon(\Sigma) \geq |\mathcal{C}|$.

Lemma 1.4.26 ([30, Lemma 4.1.2])**.** Let P_1, P_2, P_3 be three arcs, between the same two endpoints but otherwise disjoint.

(i) $\mathbb{R}^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly three regions, with frontiers $P_1 \cup P_2, P_2 \cup P_3$ and $P_1 \cup P_3$.

(ii) If P is an arc between a point in $\mathring{P}_1$ and a point in $\mathring{P}_3$ whose interior lies in the region $\mathbb{R}^2 \setminus (P_1 \cup P_3)$ that contains $\mathring{P}_2$, then $\mathring{P} \cap \mathring{P}_2 \neq \emptyset$.

CHAPTER 2

BASICS OF THE WELL-QUASI-ORDERING THEORY

As remarked by Luscanne [89], the concept of well-quasi-ordering goes back at least in the begging of the 20th century, since among the forerunners is Janet [65] whose paper appeared in 1920. Jean H. Gallier notes in [52] that Irving Kaplanski told him that this notion is defined and used in his Ph.D thesis [68] in 1941, in which unfortunately we couldn't access, that's why we mention Gallier's reference. However, a clear evidence which places Irving Kaplanski among the forerunners of the well-quasi-ordering theory is an exercise which he suggests in textbook of 1948 [14, Exercise 8, p.39]. In this exercise, he claims the equivalence of conditions (ii) and (iii) of the characterizations of the well-quasi-order notion given by Theorem 2.1.7. Richard Rado in 1954 [100], appears to be aware of this theorem that Kaplanski proposed as an exercise.

Joseph Kruskal in an extended historical recursion and general presentation of the well-quasi-ordering concept in [82] -which we suggest as a further reading- notes that notions-forerunners of the well-quasi-ordering concept can be found at [84], in which Georges Kurepa in 1937 invented a concept which is closely related to well-quasi-ordering. He also mentions as forerunners the conjecture for finite trees that Andrew Vazsonyi made in 1937, which we discuss in Section 3.2 and a problem proposed by Paul Erdős [45] in 1949, which we discuss in Section 2.5.

The aforementioned Erdős' problem appears to play a crucial role to the well-quasi-ordering concept since the first clear uses (but under different names) and theorems considering the well-quasi-ordering notion appeared in papers which are closely related to the solution of this problem.

Particularly, Paul Erdős and Richard Rado in 1952 provided a solution to the Erdős' problem in [44]. In a note at the end of this paper (see [45, p.256-257]) they define the notion of a partially well-ordered set (see Definition 2.1.3) which is equivalent to the notion of a well-quasi-ordered set and they state without proof a result which says essentially that the set of finite subsets of a partially well-ordered set is also partially well-order (this is Theorem 2.5.5). They also state that G. Higman and B.H. Neumann, independently of each other and of themselves, proved essentially that the set of finite sequences from elements of a partially well-ordered set is partially well-ordered (this is Theorem 2.6.4).

In the same year Graham Higman [64] became the first (known to us) who clearly developed the theory of well-quasi-ordering. In [64] Higman proved a theorem which has as a corollary the Erdős' problem and defined the "finite basis property" of a set (see Definition 2.1.5), which is equivalent with the property of a set to be well-quasi-ordered and proved several theorems for spaces with the finite basis property, most of these theorems are presented in this chapter.

In the introduction of [64] Higman notes that the "finite basis property" is equivalent with the "partial well-orders" of Paul Erdős and Richard Rado and he thanks them for letting him see an unpublished manuscript of them, [42] which -according to Kruskal- was probably an early version of [100] or [44]. Thus Higman was aware of the "partial well-orders" when he was writing [64].

The theory was further developed in 1954 under the name "partial well-orders" by Rado [100]. Finally, -as far as we know- the first one to use the name "well-quasi-order" was Josheph Kruskal in 1960 in [81] where he proved Vazsonyi's conjecture. The theory was further developed by Nash-Williams [95, 97] who also defined the concept of "better-quasi-orders", which is even complicated to define and thus -since we will not use it- we just mention.

Well-quasi-orders have also several applications in computer science and algorithms, which we do not mention here.[1]

In this section we prove several characterizations of the well-quasi-order notion, and theorems considering well-quasi-ordered spaces.

2.1 Definition and characterizations of the well-quasi-order notion

Definition 2.1.1 (good and bad sequences)**.** Let $\leq$ be a quasi-order on a set X, given a sequence $(x_i)_{i\in\mathbb{N}}$ on X a pair (x_i, x_j) of its terms will be said to be a *good pair* of $(x_i)_{i\in\mathbb{N}}$ if and only if $(i < j) \wedge (x_i \leq x_j)$. The sequence itself will be said to be a *good sequence* -w.r.t. $\leq$- if and only if it has at least a good pair, that is, $(\exists i, j \in \mathbb{N})[(i < j) \wedge (x_i \leq x_j)]$. A *bad sequence* is a sequence that is not good.

Definition 2.1.2 (well-quasi-order)**.** Let $\leq$ be a quasi-order on a set X. The quasi-order $\leq$ will be said to be a *well-quasi-order* on X if and only if every infinite sequence on X is good (w.r.t. $\leq$).

Definition 2.1.3 (partial well-order)**.** Let $\leq$ be a quasi-order on a set X. The quasi-order $\leq$ will be said to be a *partial well-order* on X if and only if every nonempty subset of X has at least one but no more than a finite number of (non equivalent) minimal elements (w.r.t. $\leq$).

Definition 2.1.4 (closure of a set, closed set, open set)**.** Let X be a nonempty set and $\leq$ be a quasi-order on X. Given a set $A \subseteq X$, we define as the *closure* of A -denoted by $cl(A)$- the set $\{x \in X | (\exists y \in A)[y \leq x]\}$. A set will be said to be *closed* if and only if it is equal with its closure. A set will be said to be *open* if and only if its complement is a closed set.

Definition 2.1.5 (finite basis property)**.** Let $\leq$ be a quasi-order on a set X. The quasi-ordered set X will be said to has the *finite basis property* if and only if every closed subset of X is the closure of a finite set.

Observation 2.1.6. Let X be a nonempty set and $\leq$ be a quasi-order on X. Then the union of closed subsets of X is a closed subset of X.

Theorem 2.1.7 (characterizations of the well-quasi-order notion)**.** Let $\leq$ be a quasi-order on a set X, then the following conditions are equivalent:

(i) the quasi-order $\leq$ is a well-quasi-order;

[1] see e.g. [29, 60]

(ii) every infinite sequence on X has an infinite increasing subsequence;

(iii) the set X contains nor an infinite antichain neither an infinite strictly decreasing sequence;

(iv) every quasi-order that extends $\leq$ (including $\leq$ itself) is well-founded;

(v) every nonempty subset of X has at least one but no more than a finite number of (non-equivalent) minimal elements, that is, $\leq$ is partial well-order;

(vi) every closed subset of X is the closure of a finite set, that is, X has the finite basis property;

(vii) there exists no infinite strictly increasing (w.r.t. inclusion) sequence of closed subsets of X;

(viii) there exists no infinite strictly decreasing (w.r.t. inclusion) sequence of open subsets of X.

Proof. (i) $\Rightarrow$ (ii): Let $(a_n)_{n\in\mathbb{N}}$ be an arbitrary but fixed sequence on X. We call a term a_m of the sequence $(a_n)_{n\in\mathbb{N}}$ *terminal* if there is no $n > m$ such that $a_m \leq a_n$. The number of terminal members of $(a_n)_{n\in\mathbb{N}}$ should be finite, since otherwise by choosing them in the series that they appears in $(a_n)_{n\in\mathbb{N}}$ we would form a bad sequence (w.r.t $\leq$) contradicting to our assumption that the relation $\leq$ is a well-quasi-order on X. Therefore there exist a $n_0 \in \mathbb{N}$ such that $(\forall n\mathbb{N})[n \geq n_0 \Rightarrow (\exists m \in \mathbb{N})[(m > n) \wedge (a_n \leq a_m)]$
We now proceed to the inductive construction of an infinite increasing subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ as follows:

Induction Basis: $k_1 = n_0$
Induction Hypothesis: We have choose natural numbers $k_1 < \ldots < k_{n-1}$ such that $(\forall i \in \{1, n-2\})[a_{k_i} \leq a_{k_{i+1}}]$.
Induction Step: We are choosing as k_n the least natural number such that k_n is greater that k_{n-1} and $a_{k_{n-1}} \leq a_{k_n}$. Observe that such a choice is possible since provided by our induction hypothesis that $k_{n-1} > n_0$ it follows that $a_{k_{n-1}}$ in not terminal.
Induction Conclusion: $(a_{k_n})_{n\in\mathbb{N}}$ is an increasing subsequence of $(a_n)_{n\in\mathbb{N}}$.

Since $(a_n)_{n\in\mathbb{N}}$ was an arbitrary infinite sequence in X, condition (ii) follows.

(ii) $\Rightarrow$ (iii): This is immediate.

(iii) $\Rightarrow$ (iv): Towards a contradiction, let suppose that $\leq^{'}$ is a quasi-order extending $\leq$ and $leq^{'}$ is not well-founded. Let $(a_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence which witnesses that $leq^{'}$ is not well-founded. We now distinguish the following two cases:
Case 1: Either there are infinite elements in the sequence $(a_n)_{n\in\mathbb{N}}$ related under $\leq$ and hence we can derive from $(a_n)_{n\in\mathbb{N}}$ an infinite strictly decreasing sequence on X (w.r.t $\leq$) contradicting to (iii);
Case 2: either, infinitely many elements of $(a_n)_{n\in\mathbb{N}}$ are incomparable (w.r.t $\leq$) between them. Then the set of those incomparable elements forms an infinite antichain on X which again contradicts to (iii). Thus condition (iv) follows.

(iv) $\Rightarrow$ (v): Let $A \subseteq X$ be a nonempty set, since the relation $\leq$ is well-founded it follows that the set A has at least one minimal element. Towards a contradiction we suppose that the set A has infinitely many non-equivalent minimal elements, let $\{a_1, a_2, \ldots\}$ be the set of those elements. Consider now the relation $\leq^{'} = \leq \cup\{(a_i, a_j)|(j \in \mathbb{N}) \wedge (i = j + 1)\}$. The relation $\leq^{'}$ is an extension of the relation leq which is not well-founded because the sequence $(a_n)_{n\in\mathbb{N}}$ is an infinite decreasing sequence (w.r.t $\leq^{'}$), We have thus derive a contradiction and hence the condition (v) follows.

(v) ⇒ (vi): Let $A \subseteq X$ be a closed set. If $A = \emptyset$, note that $cl(\emptyset) = \emptyset$, hence A is the closure of $\emptyset$ which is a finite set. If $A \neq \emptyset$, then the set A has a finite -but no zero- number of non-equivalent minimal elements. Let n be the number of those elements and let the set $\{a_1, \ldots, a_n\}$ be a set of n non-equivalent elements of A. Since $\{a_1, \ldots, a_n\} \subseteq A$, by the definition of the closure of a set it follows that $cl(\{a_1, \ldots, a_n\}) \subseteq cl(A)$. Let a be an arbitrary but fixed element of the set A, then $(\exists i \in \{1, \ldots, n\})[a_i \leq a]$ and thus $a \in cl(\{a_1, \ldots, a_n\})$. Since a was an arbitrary element of the set A, it follows that $A \subseteq cl(\{a_1, \ldots, a_n\})$. By our assumption that the set A is a closed set, we have $A = cl(A)$. Thus we conclude that $A = cl(\{a_1, \ldots, a_n\})$, and hence indeed the set A is the closure of a finite set. As the set A was an arbitrary set, (vi) follows.

(vi) ⇒ (vii): Toward a contradiction we suppose that there exist an infinite strictly increasing sequence of closed subsets of X. Let $(A_n)_{n\in\mathbb{N}}$ be such a sequence. Consider the set $A = \bigcup_{i\in\mathbb{N}} A_i$, by Observation 2.1.6, it follows that the set A is closed and thus it is equal with a closure a finite set let $B = \{x_1, \ldots, x_n\} \subseteq X$ be such a set. Since $A = cl(\{x_1, \ldots, x_n\})$, it follows that $\{x_1, \ldots, x_n\} \subseteq A$ and thus $(\forall i \in \{1, \ldots, n\})(\exists j \in \mathbb{N})[x_i \in A_j]$. Since the sequence $(A_n)_{n\in\mathbb{N}}$ is strictly increasing it follows that there exist $j \in \mathbb{N}$ such that $\{x_1, \ldots, x_n\} \subseteq A_j$ but this implies that $cl(\{x_1, \ldots, x_n\}) \subseteq A_j$ and thus $cl(\{x_1, \ldots, x_n\}) = A_j$. Hence, $A = A_j$ which contradicts with our assumption that $(A_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence. Thus no such a sequence exists and (vii) follows.

(vii) ⇔ (viii): Just observe that $(A_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of closed sets if and only if $(X \setminus A_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence of open sets.

(viii) ⇒ (i): Since (vii) ⇔ (viii), it sufficient to prove that (vii) ⇔ (i). Towards a contradiction we suppose that the condition (vi) holds but the relation $\leq$ is not a well-quasi-order on the set X and let $(x_n)_{n\in\mathbb{N}}$ be a sequence which witness that, i.e. $(x_n)_{n\in\mathbb{N}}$ is a bad sequence. Thus $(\forall i \in \mathbb{N})(\forall j \in \mathbb{N})[i < j \Rightarrow x_i \nleq x_j]$ and hence it is immediate that the sequence $cl(\{x_1\}) \subset cl(\{x_1, x_2\}) \subset cl(\{x_1, x_2, x_3\}) \ldots$ is a strictly increasing sequence of closed subset of X, contradicting to our assumption thus the relation $\leq$ is indeed a well-quasi-order on X and the condition (i) holds. □

Comment 2.1.8. *The analogue of the characterization that is given by condition (v) of Theorem 2.1.7 for the well-quasi-order notion with the definition of the well-founded binary relations (Definition 1.1.41) "justifies" the existence of the adjective well in the name «well-quasi-order».*

It is interesting to observe that the property of being well-quasi-order for a binary relation is stronger than the property of being well-founded. Indeed, it is not true in general that any quasi-order that extends a given well-founded quasi-order is well-founded, however condition (iv) of Theorem 2.1.7, indicates that this property characterizes every well-quasi-order.

2.2 Subsets and images via order homomorphisms

Theorem 2.2.1. Let X be a set, $\leq$ be a well-quasi-order on X, Y a quasi-ordered set by a relation $\preceq$ and $f : X \to Y$ be an order homomorphism. Then, for every $A \subseteq X$ the set A is well-quasi-ordered by the relation $\leq$ and the set $f(A)$ is well-quasi-ordered by the relation $\preceq$.

Proof. The fact that every subset of a well quasi-ordered set is well-quasi-ordered by the same relation with its superset, is immediate from the definition of the well-quasi-order notion.

For the images of subsets of X via an order homomorphism, let A be an arbitrary but fixed subeset of X and observe that if $(y_n)_{n\in\mathbb{N}} \subseteq f(A)$ is an infinite bad sequence (w.r.t $\preceq$) on $f(A)$, then the sequence $(f^{-1}(y_n))_{n\in\mathbb{N}}$ is an infinite bad sequence (w.r.t. $\leq$) on A and thus on X, which is an absurd. Hence there exist no bad sequence on $f(A)$ and thus $f(A)$ is well-quasi-ordered by the relation $\preceq$, since A is was an arbitrary subset of X the theorem follows. ☐

2.3 Induction schemes for well-quasi-ordered spaces

In this section we prove two induction schemes for well-quasi-ordered spaces. The first is an implication of the characterization of the well-quasi-order notion by condition (iii) of Theorem 2.1.7 and shows that we can induct over the elements of a well-quasi-ordered space and the second follows from condition (viii) of Theorem 2.1.7 and shows that we can induct over the open sets of a well-quasi-ordered space.[64, Theorem 2.4]

Notation 2.3.1. Let X be a set and Π be a statement for the elements of X. If $x \in X$, we denote by $\Pi(x)$ the fact that the statement Π is true for the element x.

Theorem 2.3.2 (Induction scheme I)**.** Let X be a set that is well-quasi-ordered by a relation $\leq$, and let Π be a statement for the elements of X, such that:

(i) If x is a minimal element of X, then $\Pi(x)$;

(ii) For all $x \in X$, $\Pi(x)$ provided that $\Pi(y)$ for all $y < x$.

Then $(\forall x \in X)[\Pi(x)]$.

Proof. Towards a contradiction we suppose that the statement Π is not true $\forall x \in X$. We now proceed to the inductive construction of an infinite strictly decreasing sequence of X, as follows:

Induction Basis: We are choosing an element $x_0 \in X$, such that the statement Π is not true for x_0.

Induction Hypothesis: We suppose that we have chosen n elements of the set X, such that $x_0 > \ldots > x_{n-1}$ and the statement Π is not true for the element x_{n-1}.

Induction Step: From our assumptions $(i), (ii)$ for the statement Π, it follows that x_{n-1} is not a minimal element of X and that there exist $x_n \in x$ such that $x_{n-1} > x_n$ and the statement Π is not true for x_n. We are choosing x_n as the $(n+1)$ term of our sequence.

Induction Conclusion: For each $n \in \mathbb{N}$ we have chosen an element x_n, such that $x_n > x_{n+1}$. Hence, we have construct the desired sequence.

Thus by assuming that the statement Π is not true $\forall x \in X$, we have construct an infinite strictly decreasing sequence on X which by Theorem 2.1.7 contradicts to the fact that X is well-quasi-ordered. Hence our assumption was false and the proof is complete. ☐

Notation 2.3.3. Let X be a set and Π be a statement for the subsets of X. If $A \subseteq X$, we denote by $\Pi(A)$ the fact that the statement Π is true for the set A.

Note that the emptyset is the "smallest" -with respect to set-inclusion- open set of any well-quasi-ordered space.

Theorem 2.3.4 (Induction scheme II)**.** Let X be a set that is well-quasi-ordered by a relation $\leq$, and let Π be a statement for subsets of X, such that:

(i) $\Pi(\emptyset)$;

(ii) For all open subsets $A \subseteq X$, we have $\Pi(A)$ provided that Π holds for all downward closed proper subsets $B \subset A$.

Then $\Pi(X)$.

Proof. Towards a contradiction we suppose that the statement Π is not true for the set X. We now proceed to the inductive construction of an infinite strictly decreasing chain of open subsets of X, as follows:

Induction Basis: We are choosing an open $A_0 \subsetneq X$, such that the statement Π is not true for A_0. Observe that such a choice is possible since by our assumption that the statement Π does not hold for the set X and our assumptions $(i), (ii)$ for the statement Π it follows the existence of at least one such a set.

Induction Hypothesis: We suppose that we have chosen n open subsets of X, such that $A_0 \supsetneq \ldots \supsetneq x_{n-1}$ and the statement Π is not true for the open set A_{n-1}

Induction Step: From our assumptions $(i), (ii)$ for the statement Π, it follows that A_{n-1} is not the empty set and that there exist an open $A_n \subset X$ such that $A_{n-1} \supsetneq A_n$ and the statement Π is not true for A_n. We are choosing A_n as the $(n+1)$ term of our sequence.

Induction Conclusion: For each $n \in \mathbb{N}$ we have chosen an open set A_n, such that $A_n \supsetneq A_{n+1}$. Hence, we have construct the desired chain.

Thus by assuming that the statement Π is not true for the set X, we have construct an infinite strictly decreasing chain of open subsets of X which by Theorem 2.1.7 contradicts to the fact that X is well-quasi-ordered. Hence our assumption was false and the proof is complete. □

2.4 Nash-Williams's minimal bad sequence argument

The minimal-bad-sequence argument was first used by Nash-Williams [95] in his proof of the well-quasi-ordering of finite trees by the topological minor relation which we present in Subsection 3.2.2.

Definition 2.4.1 (minimal bad sequence)**.** Let $\leq$ be a quasi-order on a set X and $(x_n)_{n\in\mathbb{N}}$ be a bad sequence on X. We say that $(x_n)_{n\in\mathbb{N}}$ is a *minimal bad sequence* on X if and only if for each $n \in \mathbb{N}$ the element x_n of X is minimal (w.r.t $\leq$) such that a bad sequence of X has $x_0, \ldots, x_n$ as its initial segment, i.e. for each $n \in \mathbb{N}$ and each $y \in X$ with $y \leq x_n$ there is no bad sequence of X that has $x_1, \ldots, x_{n-1}, y$ as its initial segment.

Lemma 2.4.2. Let X be a set, let also $\leq$ be a quasi-order on X that is well-founded but is a not well-quasi-order in X, then there exist a minimal bad sequence on X.

Proof. We proceed to the inductive construction of a minimal bad sequence $(x_n)_{n\in\mathbb{N}}$ in X as follows:

Induction Basis: We are choosing x_0 to be a minimal (w.r.t $\leq$) element of X such that a bad sequence on X has this element as its first term. Observe that such a choice is possible since by our assumption that X is not well-quasi-ordered it follows that X has at least one bad sequence, and by our assumption that the relation $\leq$ is well founded, it follows that the set of the first terms of all bad sequences of X has at least one minimal element.

Induction Hypothesis: We assume that we have choose elements $x_0, \dots, x_{n-1}$ such that $x_0, \dots, x_{n-1}$ is an initial segment of a bad sequence in X and the element x_{n-1} is minimal (w.r.t $\leq$) such that a bad sequence in X has $x_0, \dots, x_{n-1}$ as its initial segment.

Induction Step: We are choosing x_n to be a minimal (w.r.t $\leq$) element of X, such that $x_0, \dots, x_n$ is an initial segment of a bad sequence in X. Again such a choice is possible since from the induction hypothesis there exist at least one bad sequence in X with $x_0, \dots, x_{n-1}$ as its initial segment and from the our assumption that $\leq$ is a quasi-order on X it follows that the set of nth terms of all bad sequenced of X that have $x_0, \dots, x_{n-1}$ as their initial segment has at least one minimal element.

Induction Conclusion: Obviously $(x_n)_{n\in\mathbb{N}}$ is a bad sequence in X and $\forall n \in \mathbb{N}$ if $(y_n)_{n\in\mathbb{N}}$ is a bad sequence in X such that $y_i = x_i, \forall i \in \{0, \dots, n-1\}$ then $y_n \not\lneq x_n$. □

2.5 Finite sets: Erdős & Rado's theorem

Paul Erdős in [45, Problem 4358] proposed as a problem the proof of the following theorem:

Theorem 2.5.1. If a set X of positive integers does not contain any infinite subset no element of which divides any other element, then neither does $\Pi(X)$, the set of integers which can be written as products of elements of X.

The assumption for the set X to the above theorem, which requires that X has no infinite antichain with respect to the divisibility relation, is equivalent with the requirement that X is well-quasi-ordered by the divisibility relation. To conclude that, recall the condition (iii) of Theorem 2.1.7 and note that no infinite strictly decreasing (w.r.t to the divisibility relation) sequence of positive integers exists.

The main result of this section, which is Theorem 2.5.5, was prooved by P. Erdős and R. Rado in [42] and used for the proof of Theorem 2.5.1.

Notation 2.5.2. Given a set X we will denote by $[X]^k$ the set of all subsets of X consisting of exactly k elements. With $[X]^{<\omega}$ we will denote the set of all finite subsets of X.

Below we define a quasi-order on the set of finite subsets of a quasi-ordered set.

Definition 2.5.3. Given a set X that is quasi-ordered by the relation $\leq$. We define the relation $\sqsubseteq$ on $[X]^{<\omega}$ as follows: For two finite subsets A, B of X, $A \sqsubseteq B$ if and only if there exist an injective mapping $f : A \to B$ such that $(\forall a \in A)[a \leq f(a)]$.

Observation 2.5.4. Given a set X that is quasi-ordered by the relation $\leq$, the set $[X]^{<\omega}$ is quasi-ordered by the relation $\sqsubseteq$, as the latter defined above: For the reflexivity of $\sqsubseteq$, observe that if $A \in [X]^{<\omega}$, then identity function $I : A \to A$ witness that $A \sqsubseteq A$. For the transitivity of $\sqsubseteq$, let $A, B, C \in [X]^{<\omega}$ such that $A \sqsubseteq B$ and $B \sqsubseteq C$, let also f, g be the functions that witness those relations respectively, then the function $f \circ g$ witness that $A \sqsubseteq C$.

Theorem 2.5.5. Let X be a set and $\leq$ be a binary relation on X. If $\leq$ is a well-quasi-order on X, then $\sqsubseteq$ is a well-quasi-order on $[X]^{<\omega}$.

Proof. Towards a contradiction we suppose that $\leq$ is a well-quasi-order on X, but $\sqsubseteq$ is not a well-quasi-order on $[X]^{<\omega}$. Then by Lemma 2.4.2 there exist a minimal (w.r.t the order of its elements) bad sequence, say $(A_n)_{n\in\mathbb{N}}$ on $[X]^{<\omega}$. Since $(A_n)_{n\in\mathbb{N}}$ is a bad sequence in $[X]^{<\omega}$, it follows that $A_n \neq \emptyset$, $\forall n \in \mathbb{N}$, otherwise we could find infinite good pairs in $(A_n)_{n\in\mathbb{N}}$ since $\emptyset \sqsubseteq A_n$, $\forall n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we are choosing an element $a_n \in A_n$ and considering the set $B_n := A_n \setminus \{a_n\}$. Since $(a_n)_{n\in\mathbb{N}} \subseteq \bigcup_{n\in\mathbb{N}} A_n \subseteq X$, from our hypothesis that X is well-quasi-ordered and Theorem 2.1.7 it follows that the sequence $(a_n)_{n\in\mathbb{N}}$ has an increasing subsequence. Let $(a_{n_i})_{i\in\mathbb{N}}$ be such a subsequence. Let $n_0 \in \mathbb{N}$ be an arbitrary but fixed natural number, from the minimality of $(A_n)_{n\in\mathbb{N}}$ and the fact that $|B_n| < |A_n|$, $\forall n \in \mathbb{N}$ it follows that the sequence:

$$A_0, \ldots, A_{n_0-1}, B_{n_0}, B_{n_1}, B_{n_2}, \ldots$$

is a good sequence and hence, has at least one good pair.

Let us consider such a pair. Since $(A_n)_{n\in\mathbb{N}}$ is a bad sequence the pair cannot have the form (A_i, A_j), but neither the form (A_i, B_j) because $B_j \sqsubseteq A_j$. Therefore we conclude that it has the form (B_i, B_j). Extending the injection that witness the relation $B_i \sqsubseteq B_j$ to map the element a_i to the element a_j we deduce an injection that witness $A_i \sqsubseteq A_j$ and thus that the pair (A_i, A_j) is a good pair of the bad sequence $(A_n)_{n\in\mathbb{N}}$, which is a contradiction. Hence our assumption was false, the set $[X]^{<\omega}$ is indeed well-quasi-ordered by the relation $\sqsubseteq$ and proof of the theorem is complete. □

2.6 Finite sequences & finite words: Higman's theorem

Notation 2.6.1. Let X be a set, we denote by $V(X)$ the set of all finite sequences of elements of X. We denote by ε the empty sequence, which is trivially a finite sequence of X.

Below we define a quasi-order on the set of finite sequences which are formed from elements of a quasi-ordered set.

Definition 2.6.2. Given a set X that is quasi-ordered by the relation $\leq$. We define the relation $\sqsubseteq$ on $V(X)$ as follows: For two finite sequences $x = (x_i)_{i=1}^n, y = (y_i)_{i=1}^m \in V(X)$ of X, $x \sqsubseteq y$ if and only if there exist a strictly increasing function $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $(\forall i \in \{1, \ldots, n\})[x_i \leq y_{f(i)}]$.

Observation 2.6.3. Given a set X that is quasi-ordered by the relation $\leq$, the set $V(X)$ is quasi-ordered by the relation $\sqsubseteq$, as the latter defined above: For the reflexivity of $\sqsubseteq$, observe that if $x = (x_i)_{i=1}^n \in V(X)$, then the the identity function $I : \{1, \ldots, n\} \to \{1, \ldots, n\}$ witness that $x \sqsubseteq x$. For the transitivity of $\sqsubseteq$, let $x, y, z \in V(X)$ such that $x \sqsubseteq y$ and $y \sqsubseteq z$, let also f, g be the functions that witness those relations respectively, then the function $f \circ g$ witness that $x \sqsubseteq z$.

Theorem 2.6.4 (Higman [64, Theorem 4.3])**.** Let X be a set and $\leq$ be a quasi-order on X. If $\leq$ is a well-quasi-order on X, then $\sqsubseteq$ is a well-quasi-order on $V(X)$.

Theorem 2.6.5 (Higman [64, Theorem 4.4])**.** If X is an set of words formed from a finite alphabet, it is possible to find a subset X_0 of X such that, given a word $w \in X$, it is possible to find a word $w_0 \in X_0$ such that the letters of w_0 occur in w in their right order, though not necessarily consecutively.

2.7 Finite cartesian product: Higman's theorem

Definition 2.7.1. Let X, Y be two sets that are quasi-ordered by the relations $\leq_1, \leq_2$ respectively. We define the relation $\sqsubseteq$ on the cartesian product of X, Y as follows: $\sqsubseteq = \{((x_1, y_1), (x_2, y_2)) | (x_1 \leq_1 x_2) \wedge (y_1 \leq_2 y_2)\}$.

Observation 2.7.2. Given two quasi-ordered sets, the relation $\sqsubseteq$ on their cartesian product as defined above is a quasi-order on it.

The following was proved by Higman [64, Theorem 2.3].

Theorem 2.7.3. Let X, Y be two sets that are well-quasi-ordered by the relations $\leq_1, \leq_2$ respectively, then their cartesian product $X \times Y$ is well-quasi-ordered by the relation $\sqsubseteq$.

Proof. Let $((x_n, y_n))_{n\in\mathbb{N}}$ be an arbitrary but fixed infinite sequence on $X \times Y$. From our assumption that X is well-quasi-ordered and Theorem 2.1.7, it follows that the infinite sequence $(x_n)_{n\in\mathbb{N}}$ has an infinite increasing (w.r.t. $\leq_1$) subsequence, let $(x_{k_n})_{n\in\mathbb{N}}$ be such a sequence. For the same reasons the infinite sequence $(y_{k_n})_{n\in\mathbb{N}}$ has an infinite increasing (w.r.t. $\leq_2$) subsequence, say $(y_{k_{l_n}})_{n\in\mathbb{N}}$. Then the sequence $(x_{k_{l_n}}, y_{k_{l_n}})_{n\in\mathbb{N}}$ is an infinite increasing (w.r.t. $\sqsubseteq$) subsequence of $((x_n, y_n))_{n\in\mathbb{N}}$. Since $((x_n, y_n))_{n\in\mathbb{N}}$ was an an arbitrary infinite sequence on $X \times Y$, by Theorem 2.1.7 follows that the set $X \times Y$ is well-quasi-ordered by the relation $\sqsubseteq$. □

Definition 2.7.4. Let $X_1, \ldots X_n$ be n sets that are quasi-ordered by the relations $\leq_1, \ldots, \leq_n$ respectively. We define the relation $\sqsubseteq$ on the cartesian product of $X_1, \ldots, X_n$ as follows: $\sqsubseteq = \{\big((x_1, \ldots, x_n), (y_1, \ldots, y_n)\big) | (x_1 \leq_1 y_1) \wedge \ldots \wedge (x_n \leq_n y_n)\}$.

Observation 2.7.5. Given n quasi-ordered sets, the relation $\sqsubseteq$ on their cartesian product as defined above is a quasi-order on it.

By induction and using Theorem 2.7.3 for the induction base and the induction step we get the following:

Theorem 2.7.6. Let $X_1, \ldots X_n$ be n sets that are quasi-ordered by the relations $\leq_1, \ldots, \leq_n$ respectively, then their cartesian product $X_1, \ldots, X_n$ is well-quasi-ordered by the relation $\sqsubseteq$.

2.8 An application of well-quasi-ordering theory on graphs

Theorem 2.8.1. Let $\mathcal{X}$ be a set of graphs that is well-quasi-ordered by a relation $\leq$ and let $\mathcal{Q} \subseteq \mathcal{X}$ be a property of graphs that is closed under the relation $\leq$. Then there exist an integer k (only depending on $\mathcal{Q}$) and graphs $H_1, H_2, \ldots, H_k$ such that an arbitrary graph $G \in \mathcal{X}$ satisfies $\mathcal{Q}$ if and only if $(\forall i \in \{1, 2, \ldots, k\})[H_i \nleq G]$.

Proof. Let $\mathcal{F}$ be the complement of $\mathcal{Q}$ on $\mathcal{X}$. Notice that $\mathcal{F}$ is a closed set and hence by Theorem 2.1.7 there exist a finite set of graphs $\mathcal{H}$ such that $\mathcal{F} = cl(\mathcal{H})$. The set $\mathcal{H}$ is the desired set of graphs and the desired integer k is the cardinality of $\mathcal{H}$. □

CHAPTER 3

WELL-QUASI-ORDERING GRAPHS BY THE MINOR RELATION

Robertson and Seymour's theorem states that graphs are well-quasi-ordered by the minor relation. In this chapter we present -relatively short- proofs of several special cases of Robertson and Seymour's theorem. In particular, we prove that trees, graphs of bounded branch-width (similarly of bounded tree-width), planar graphs and graphs which exclude a fixed planar graph as a minor are well-quasi-ordered by the minor relation. We also give a direct proof of perhaps the most interesting special case of Robertson and Seymour's theorem which states that embeddability in any fixed surface can be characterized by forbidding finitely many minors. In order to derive some of our results we introduce the notions of graph's branch-decomposition and branch-width and we prove a Menger-like property of branch-decompositions. For the same purpose we also prove two "structure theorems" which characterize the "rough" structure of a planar graph which exclude a fixed grid as a minor and the "rough" structure of an arbitrary graph which does so. Since most of the results presented in this chapter are cornerstones of Neil Robertson and Paul Seymour's Graph Minors series [104] we begin by presenting the main motivations of their work and by illustrating the interplay between "structure theorems", results considering graphs' embeddings and well-quasi-ordering theorems in their approach to Wagner's conjecture.

3.1 Introduction

In their Graph Minors series Robertson and Seymour [104] among other great results proved (in [114]) Wagner's conjecture, today known as the *Robertson and Seymour's theorem*. Robertson and Seymour's theorem states that in any infinite set of graphs there exist two such that the one is isomorphic to a minor of the other, since there exist no infinite strictly decreasing sequences of finite graphs with respect to the minor relation another formulation of Robertson and Seymour's theorem is that graphs are well-quasi-ordered by the minor relation. Robertson and Seymour's theorem and the notions and methods developed for its proof have been worked and is still working as a model for the study of other graphs' relations and for the development of analogue tools and methods and the deduction of analogue results for the appropriate for directed graphs notion of minor. Because of the above and the deep interplay among the areas of structural graph theory, topological graph theory and graph's well-quasi-ordering theory in the Graph Minors series we chose the special cases of Robertson and Seymour's theorem as paradigms in order to indicate this interplay. As the understanding of the results presented in this chapter is a major step in the understanding of the whole Graph Minors Project, this section intends to illustrate some answers to the following questions:

- Which theorems and conjectures motivate Robertson and Seymour's Graph Minors series?
- Which theorems and proofs' techniques regarding well-quasi-ordering played an important role in the first steps of the Graph Minors series?
- Which was Robertson and Seymour's general approach to Wagner's conjecture and which proofs' techniques worked as a paradigm for it?
- How does structure theorems, theorems regarding graphs' embeddings and well-quasi-ordering theorems are connected each other?
- Which are the most important results of the Graph Minors series?

Definitely our starting point in order to answer the first question has to be the characterization of planar graphs in terms of forbidden topological minors (Theorem 3.1.1) by Kazimierz Kuratowski [83] in 1930, which is one of the most famous results in graph theory. This theorem where previously proved -although never published- around 1927 by Lev Semyonovich Pontryagin, and at the same year with Kuratowski's publication[1] by Orrin Frink & Paul Smith[2].

Theorem 3.1.1 (Pontryagin - Kuratowski [83] - Orrin Frink & Paul Smith [51])**.** A graph is planar if and only if it has no topological minor isomorphic to K_5 or $K_{3,3}$.

In 1937 Klaus Wagner [125] restated and proved the above characterization of planar graphs in terms of minors instead of topological minors.

Theorem 3.1.2 (Wagner [125])**.** A graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.

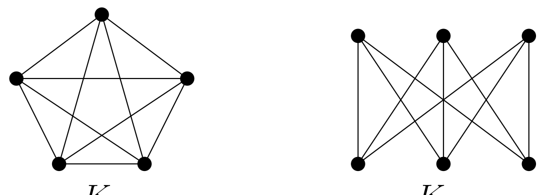

Figure 3.1.1: Forbidden minors for planar graphs.

Recall from Definition 1.2.60 that a property of graphs, say $\mathcal{M}$, is said to be minor-closed if and only if $(\forall G \in \mathcal{M})[H \leq_m G \to H \in \mathcal{M}]$. Given a minor-closed property of graphs, say

[1] But independently of it.

[2] For more details on the history of the theorem which characterize the planar graphs we refer the interested reader in [72].

$\mathcal{M}$, this property can be characterized by the set of all minor-minimal graphs which are not in $\mathcal{M}$. This set is called the set of *forbidden minors*, or the *Kuratowski set* of $\mathcal{M}$. We remark that the set of forbidden minors for every minor-closed property of graphs is an antichain with respect to the minor relation. Graph properties which are minor-closed occur frequently in graph theory. For example embeddability in any fixed surface is such a property. Wagner's reformulation of Kuratowski's theorem is the first theorem which characterize a -non-trivial- graph property which is closed under taking minors by a set of forbidden minors. Another such characterization of a minor-closed graph property was given by Dirac [38] in 1952, who proved the following theorem:

Theorem 3.1.3 (Dirac [38])**.** A graph is series-parallel if and only if has no minor isomorphic to K_4.

An immediate and natural question which arise from the above, is if a similar with Kuratowski-Wagner's result holds for other surfaces and more generally if there exist such characterizations for other minor-closed graph properties.

In 1930's Paul Erdős and Denes König conjectured that embeddability in any fixed surface can be characterized by forbidding finitely many graphs as topological minors and thus it is a generalization of Kuratowski's theorem. Robertson and Seymour [110] indicate Erdős as responsible for the conjecture, Thomassen [122] indicates both Erdős and König, finally Bodendiek and Wagner [16] indicate only König. We only found the conjecture written, by König, and for the case of orientable surfaces, in the first -ever published- book considering graph theory, which was written by König [76, see at the top of page 199] in 1936.

Conjecture 3.1.4 (Paul Erdős and Denes König [76])**.** For any surface Σ, there exist a positive integer n and graphs $G_1, \ldots, G_n$ such that an arbitrary graph G is embeddable in Σ if and only if G has no topological minor isomorphic which is isomorphic to a graph in $\{G_1, \ldots, G_n\}$.

A constructive proof for the case of non-orientable surfaces was given by Archdeacon and Huneke [3] and a non-constructive proof for general surfaces and thus a complete proof of the conjecture was published by Robertson and Seymour [110] in 1990. Actually, Robertson and Seymour in Section 2 of [110] proved that Conjecture 3.1.4 is equivalent with the following theorem which they proved in the same paper.

Theorem 3.1.5 (Robertson and Seymour [110])**.** For any surface Σ, there exist a positive integer n and graphs $G_1, \ldots, G_n$ such that an arbitrary graph G is embeddable in Σ if and only if G has no minor isomorphic which is isomorphic to a graph in $\{G_1, \ldots, G_n\}$.

The proof by Robertson and Seymour [110] of the above theorem is long and difficult. However, there is now a remarkably accessible proof based on their original ideas which we present this proof is in Section 3.7.

In 1937 -as mentioned by Lovász [88]- Vázsonyi, made the following conjecture:

Conjecture 3.1.6 (Vázsonyi's 1937)**.** There is no infinite set $\{T_1, T_2, \ldots\}$ of -finite- trees such that T_i is not isomorphic to a topological minor of T_j for all $i \neq j$.

Since there exists no infinite strictly decreasing sequence of (finite) trees with respect to the topological minor relation Vázsonyi's conjecture is equivalent with the statement that trees are well-quasi-ordered by the topological minor relation. The conjecture proved independently in 1960 by Kruskal [81] and Tarkowski [116]. A much shorter and elegant proof was given by Nash-Williams [95] in 1963, we present Nash-Williams' proof in Subsection 3.2.2.

Theorem 3.1.7 (Kruskal [81], Tarkowski [116])**.** Trees are well-quasi-ordered by the topological minor relation.

An immediate corollary of the above theorem is the following:

Theorem 3.1.8 (Kruskal [81], Tarkowski [116])**.** Trees are well-quasi-ordered by the minor relation.

Observation 3.1.9. The set of all graphs is not well-quasi-ordered by the topological minor relation, in Figure 3.1.2 is illustrated an infinite antichain of graphs with respect to the topological minor relation.

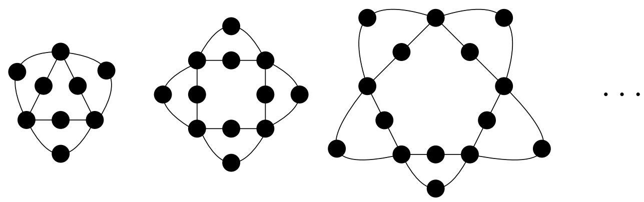

Figure 3.1.2: An infinite antichain of graphs with respect to the topological minor relation.

Probably motivated by Kuratowski's theorem, Erdős and König's conjecture, the proof of Vázsonyi's conjecture, Dirac's theorem, other similar results and the fact that the topological minor relation is not a well-quasi-order for graphs in general. Karl Wagner in 1960's and probably in [124] made the following conjecture[3]:

Conjecture 3.1.10 (Wagner [124])**.** If $G_1, G_2, \ldots$ is any infinite sequence of graphs, then there exist i, j with $j > i \geq 1$ such that G_i is isomorphic to a minor of G_j.

So, to give an answer to the first question that we set in this section, the main motivation of the Graph Minors series was Wagner's conjecture, which in its full generality was proved in Graph Minors XX [114].

Theorem 3.1.11 (Robertson ans Seymour's theorem, Robertson and Seymour [114])**.** If $G_1, G_2, \ldots$ is any infinite sequence of graphs, then there exist i, j with $j > i \geq 1$ such that G_i is isomorphic to a minor of G_j.

Although we do not deal with infinite graphs in this thesis, we remark that Robin Thomas [119] proved that Robertson and Seymour's theorem cannot generalized to uncountable graphs. The problem considering whether or not countable graphs are well-quasi-ordered by the minor relation is wide open. The following are different formulations of Robertson and Seymour's theorem[4]

[3]Robertson and Seymour always (see e.g. [104, 107]) referred to the Robertson ans Seymour's theorem as *Wagner's conjecture*. Diestel [30] notes that *«Wagner did indeed discuss this problem in the 1960s with his then students, Halin and Mader, and it seems that Mader conjectured a positive solution. Wagner himself always insisted that he did not—even after the Robertson ans Seymour's theorem had been proved»*. Lovász [88] also refer to the Robertson ans Seymour's theorem as Wagner's conjecture and he refer the same textbook [124], which Robertson and Seymour [114] refer. We remark that we couldn't access this textbook [124], but we notice that in the introduction of their paper "Solution to König's graph embedding problem", Bodendiek and Wagner [17], make direct reference to the Wagner's conjecture under the name "Wagner's well-quasi-ordering conjecture".

[4]Throughout this thesis whenever we refer to Robertson and Seymour's theorem we may refer at any of its equivalent formulations.

Theorem 3.1.12 (Robertson and Seymour's theorem, Robertson and Seymour [114])**.** Every antichain of graphs with respect to the minor relation is finite.

Theorem 3.1.13 (Robertson and Seymour's theorem, Robertson and Seymour [114])**.** Graphs are well-quasi-ordered by the minor relation.

Theorem 3.1.14 (Robertson and Seymour's theorem, Robertson and Seymour [114])**.** For every minor-closed graph class the set of forbidden minors is finite.

The latter of the above formulations of Robertson and Seymour's theorem, illustrates clearly the relation of Wagner's conjecture with Erdős and König's conjecture. The former extended the finite basis property of graphs embeddable in a fixed surface which was conjectured in the latter to arbitrary minor-closed classes of graphs.

The second main motivating problem (see [107]) for the Graph Minor series was the k-Disjoint Paths Problem, which given a graph G and k pairs of vertices of G asks whether or not there exist k mutually vertex-disjoint paths of G joining the pairs. If k is a part of the input then the above is an NP-complete[5] problem. However, for any fixed number of pairs Robertson and Seymour [112] obtained a polynomial-time algorithm.

Theorem 3.1.15 (Robertson and Seymour [112])**.** For every fixed positive integer k, there is a polynomial-time algorithm for the k-Disjoint Paths Problem. Actually the time complexity of the algorithm is $\mathcal{O}(|V(G)|^3)$, where G is the input graph.

For the question regarding the theorems and proofs' techniques of the well-quasi-ordering theory, which played an important role in the Graph Minors series, we restrict ourselves to refer the minimal bad sequence argument which Nash-Williams [95] (Lemma 2.4.2) used for the proof of the well-quasi-ordering of finite trees and Higman's [64] "finite sequences" theorem (Theorem 2.6.4). These theorems were basic ingredients for the proofs of the well-quasi-ordering of graphs of bounded tree-width [109] (Theorem 3.4.1) and of Kuratowski's theorem for general surfaces[110] (the characterization of the embeddability for any fixed surface Theorem 3.1.5).

Regarding their "structure theorem"-approach to Wagner's conjecture Robertson and Seymour refer in [107] that «the starting point for the project was Mader's use of a theorem of Erdős and Pósa». Erdős and Pósa [43] in 1965 proved the following structure theorem.

Theorem 3.1.16. [Erdős and Pósa [43]] Given a natural number k there exists a natural number k' such that for every graph G

- Either G has k vertex-disjoint cycles, or
- there exists a set of vertices $X \subseteq V(G)$, such that $|X| < k'$ and the graph $G \setminus X$ contains no cycles.

Mader [90] using the above theorem, proved the following.

Theorem 3.1.17 (Mader [90])**.** Let k be a positive integer. The set of all graphs with no k-vertex-disjoint cycles as subgraphs is well-quasi-ordered by the topological minor relation.

[5] We do not define neither the notion of NP-completeness nor other notions considering algorithms in this thesis. We refer the interest reader in any textbook of algorithms (e.g. [25])

Robertson and Seymour described their "structure theorem"-approach to Wagner's conjecture in [107] as follows:

> Let Σ be a "structure" of graphs, for example being planar, or having genus $\leq k$, or being divisible into small cutsets. Let us identify Σ with the class of graphs possesing this structure. Now suppose that we wish to show that Σ is well-quasi-ordered by minors, and suppose that we can prove a structure theorem within Σ of the following kind.
>
> For every $H \in \Sigma$ there is a structure $\Sigma'(H)$ such that $G \in \Sigma'(H)$ for every graph $G \in \Sigma$ with no minor isomorphic to H.
>
> Then it is suffices to prove that for $H \in \Sigma$, $\Sigma'(H)$ is well-quasi-ordered by minors. For if $\mathcal{C}$ is an infinite antichain with $\mathcal{C} \subseteq \Sigma$, choose $H \in \mathcal{C}$, and then $\mathcal{C} \setminus H$ is an infinite antichain in $\Sigma'(H)$.

The proofs, that planar graphs and graphs which exclude a fixed planar graph as a minor are well-quasi-ordered by the minor relation which are presented in Section 3.5 and Section 3.6 respectively, are typical applications of the above scheme.

The question regarding the interplay of graph's embeddings and well-quasi-ordering theorems in the Graph Minors series, has been partially answered by our mention to the proof [110] of the Erdős and König's conjecture which was a well-quasi-ordering result. Moreover, Robertson and Seymour need surface embeddings[6] for the proof of Wagner's conjecture in [114].

Beyond the Robertson-Seymour's theorem and the polynomial-time algorithm for the k-Disjoint Paths Problem (with k fixed), among the greatest results of the Graph Minors series are a polynomial time algorithm for testing if an arbitrary graph has a fixed graph H as a minor, the proof [103] of Nash-Williams immersion conjecture (Theorem 4.2.3) and a powerful structure theorem which captures, for any fixed graph H, the common structural features of all the graphs which do not contain a minor isomorphic to H. For the statement of the latter theorem we refer the interested reader in [71].

Theorem 3.1.18 ([112])**.** For a fixed graph H, there exist an algorithm which decides whether a given graph G contains a minor isomorphic to H in time $\mathcal{O}(|V(G)|^3)$.

The above combined with Robertson and Seymour's theorem has the following immediate corollary.

Corollary 3.1.19. For every minor-closed property $\mathcal{Q}$ of graphs there exist a polynomial-time algorithm which presented an arbitrary graph, say G, decides whether or not $G \in \mathcal{Q}$.

We suggest as further readings on the issues discussed above, the present chapter and the following [12, 32, 47, 70, 88, 102].

3.2 Trees

In this section we present the proof of Kruskal's Tree theorem which has as a corollary Vazsonyi's conjecture. Kruskal's Tree theorem states that the set of all "structured" trees that their vertices

[6] See for example the sketch of the proof of Robertson and Seymour's theorem by Diestel [32].

are labeled from a well-quasi-ordered set is well-quasi-ordered by an extension of the topological minor relation. We also give a more direct proof of Vazsonyi's conjecture in Subsection 3.2.2. Both proofs are due to Nash-Williams [95], who also proved in 1965 in [97] the well-quasi-ordering of all trees -finite and infinite- by the topological minor relation, which was a Kruskal's conjecture ([81, Conjecture 1]). We remark that Higman [64] -not in graphtheoretic-terms- proved the special case of Vazsonyi's conjecture which considers trees with bounded vertex degrees[7].

3.2.1 Vazsonyi's conjecture and Kruskal's tree theorem

In this subsection we set up some notation that we will not use further in this thesis. Some of the notation have been chosen in order to be compatible with the notation that is used in [81].

Notation 3.2.1 (the set of all finite trees)**.** We denote by $\mathcal{T}^\sharp$ the set of all finite trees.

Proposition 3.2.2. Vazsonyi's Conjecture holds if and only if the set $\mathcal{T}^\sharp$ is well-quasi-ordered by the topological minor relation.

Proof. From Theorem 2.1.7 which gives alternatives characterizations of the well-quasi-order notion, it follows that the set $\mathcal{T}^\sharp$ is well-quasi-ordered by the relation $\leq_{tm}$ if and only if the set $\mathcal{T}^\sharp$ contains nor an infinite antichain neither an infinite strictly decreasing sequence (w.r.t. $\leq_{tm}$).
Since by Observation 1.2.57 there exist no infinite strictly decreasing sequence of graphs w.r.t. the topological minor relation, it follows that the set $\mathcal{T}^\sharp$ is well-quasi-ordered by the relation $\leq_{tm}$ if and only if it does not contain an infinite antichain (w.r.t. $\leq_{tm}$) and hence $\mathcal{T}^\sharp$ is well-quasi-ordered by the relation $\leq_{tm}$ if and only if Vazsonyi's Conjecture holds. □

Definition 3.2.3 (structured tree)**.** A tree T is said to be *structured* if:

(i) T is a rooted tree, i.e a particular vertex called root of T, is specified;

(ii) every edge of T is oriented so that it points away from the root of T;

(iii) for each vertex v of T, the edges that has as tail (initial vertex) the vertex v, are linearly ordered.

Notation 3.2.4 (the set of all finite structured trees)**.** We will denote by $\mathcal{T}$ the set of all finite structured trees.

Definition 3.2.5 (monomorphism between structured trees)**.** Let $T_1, T_2 \in \mathcal{T}$. Then $\omega : V(T_1) \to V(T_2)$ is said to be a *monomorphism* if:

(i) ω is an embedding which witness that T_1 is a topological minor of T_2 when the structure on T_1 and T_2 is disregarded;

(ii) ω maps each vertex of T_1 to a vertex of T_2;

(iii) ω maps each edge of T_1 to an oriented path of T_2, and does so in an orientation preserving manner;

[7] See at the last page of [81] for what exactly Higman proved, and how it is related to Kruskal's Tree theorem

(iv) for each vertex v of T_1, ω maps the edges which has as tail the vertex v into paths which initiate with the vertex $\omega(v)$ in a manner which is strictly order-preserving (w.r.t. the linear ordering of the edges of T_1 which have as tail the vertex v and the linear ordering of the edges of T_2 which have as tail the vertex $\omega(v)$).

Definition 3.2.6 (relation $\leq_{tm}^{s}$ on $\mathcal{T}$)**.** Let $T_1, T_2 \in \mathcal{T}$, then $T_1 \leq_{tm}^{s} T_2$ if and only if there exist a monomorphism $\omega : V(T_1) \to V(T_2)$.

Definition 3.2.7 (the disregarding structure function $\sharp$)**.** We define the function $\sharp : \mathcal{T} \to \mathcal{T}^{\sharp}$ as the function that maps each structured tree T to its corresponding ordinary tree which is obtained by disregarding the structure of T.

Observation 3.2.8. Let $T_1, T_2 \in \mathcal{T}$ be two structured trees such that $T_1 \leq_{tm}^{s} T_2$, then it follows by Definition 3.2.6 that $\sharp(T_1) \leq_{tm} \sharp(T_2)$. Thus the function $\sharp$ is an order homomorphism.

Observation 3.2.9. The disregarding structure function $\sharp$ is an onto function. To see that just observe that if $T' \in \mathcal{T}^{\sharp}$ is an ordinary tree, then T' can easily transformed to a structured tree T in such a way that $\sharp(T) = T'$.

Lemma 3.2.10. If the set of all finite structured trees $\mathcal{T}$ is well-quasi-ordered by the relation $\leq_{tm}^{s}$, then the set of all finite trees $\mathcal{T}^{\sharp}$ is well-quasi-ordered by the relation $\leqq$.

Proof. It follows immediate from Observation 3.2.8, Theorem 2.2.1 and Observation 3.2.9. □

Definition 3.2.11 (structured tree over a set)**.** Let X be a nonempty set that is quasi-ordered by a relation $\leq$, let also $T \in \mathcal{T}$ be a structured tree and $t : V(T) \to X$ be a function that maps each vertex of the structured tree T to an element of the set X. We call t a *structured tree over* X Intuitively, we can visualize t as a structured tree in which each vertex is labeled with an element of the set X. We call T the *carrier* of t.

Notation 3.2.12. Given a set X, we denote by $\mathcal{T}(X)$ the set of all finite structured trees over X.

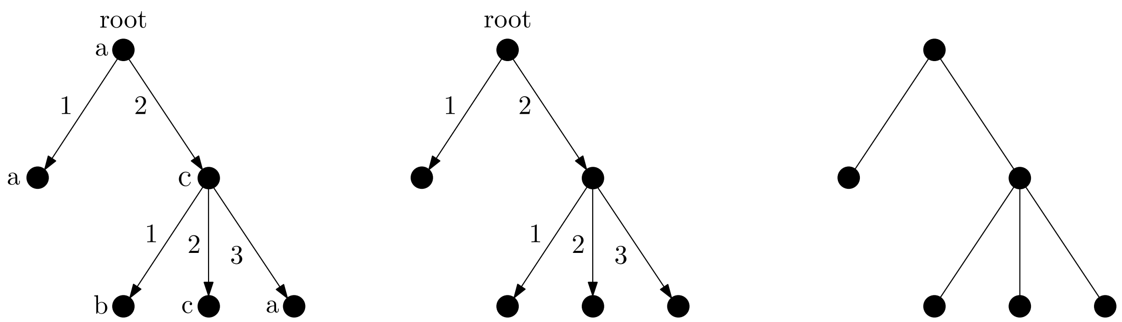

Figure 3.2.1: On the left hand side we illustrate a structured tree T over the set $\{a, b, c\}$. In the middle we have its carrier, the structured tree $\flat(T)$, and on the right hand it is illustrated the tree $\sharp(\flat(T))$ that is obtained by disregarding the structure of $\flat(T)$.

Definition 3.2.13 (monomorphism between elements of $\mathcal{T}(X)$)**.** Let X be a nonempty set that is quasi-ordered by a relation $\leq$, let also $t_1, t_2 \in \mathcal{T}(X)$ be two structured trees over X and let $T_1, T_2 \in \mathcal{T}$ be the carriers of t_1, t_2 respectively. A function $\omega : V(T_1) \to V(T_2)$ is said to be a *monomorphism* $\omega : t_1 \to t_2$ if it fulfills all the requirements of Definition 3.2.5 and also has the following additional property:

$$(\forall v \in V(T_1))[t_1(v) \leq t_2(\omega(v))].$$

Intuitively this requires that ω maps each vertex of T_1 to a vertex of T_2 with "greater" (w.r.t $\leq$) label.

Definition 3.2.14 (relation $\leq_{tm}^{sl}$ on $\mathcal{T}(X)$)**.** Let X be a nonempty set that is quasi-ordered by a relation $\leq$ and let $t_1, t_2 \in \mathcal{T}(X)$ be two structured trees over X, then $t_1 \leq_{tm}^{sl} t_2$ if and only if there exist a monomorphism $\omega : t_1 \to t_2$.

Definition 3.2.15 (unlabeling function $\flat$)**.** Let X be a nonempty set that is quasi-ordered by a relation $\leq$. We define the function $\flat : \mathcal{T}(X) \to \mathcal{T}$ as the function that maps each structured tree over X to each carrier.

Observation 3.2.16. Let X be a nonempty set that is quasi-ordered by a relation $\leq$, the unlabeling function $\flat : \mathcal{T}(X) \to \mathcal{T}$ is onto and preserves the relation $\leq_{tm}^{sl}$ i.e. is an order homomorphism.

Lemma 3.2.17. Let X be a nonempty set that is quasi-ordered by a relation $\leq$. If the set $\mathcal{T}(X)$ of the finite structured trees over X is well-quasi-ordered by the relation $\leq_{tm}^{sl}$, then the set $\mathcal{T}$ of all structured trees is well-quasi-ordered by the relation $\leq_{tm}^{s}$.

Proof. It follows immediate from Observation 3.2.16 and Theorem 2.2.1. □

$$\mathcal{T}(X) \xrightarrow{\ \flat\ } \mathcal{T} \xrightarrow{\ \sharp\ } \mathcal{T}^\sharp$$

$$X \text{ wqo} \implies \mathcal{T}(X) \text{ wqo} \implies \mathcal{T} \text{ wqo} \implies \mathcal{T}^\sharp \text{ wqo}$$

Figure 3.2.2: The tree Theorem 3.2.18

From Lemma 3.2.17 and Lemma 3.2.10 it follows immediate that given a non empty quasi-ordered set X, if $\mathcal{T}(X)$ is well-quasi-ordered by the relation $\leq_{tm}^{sl}$ then the set of all finite trees $\mathcal{T}^\sharp$ is well quasi-ordered by the topological minor relation $\leq_{tm}$ which by Proposition 3.2.2 implies the truth of Vazsonyi's Conjecture 3.1.6. Thus to prove Vazsonyi's Conjecture 3.1.6 it sufficiency to prove that there a exist a non empty quasi-ordered set X such that the set $\mathcal{T}(X)$ is well-quasi-ordered by the relation $\leq_{tm}^{sl}$. Kruskal in [81] proved the following stronger statement:

Theorem 3.2.18 (Tree Theorem)**.** Let X be a nonempty set that is quasi-ordered by a relation, say $\leq$. If X is well-quasi-ordered by the relation $\leq$, then $\mathcal{T}(X)$ is well-quasi-ordered by the relation $\leq_{tm}^{sl}$.

Proof. Towards a contradiction we suppose that the set X is well-quasi-ordered by the relation $\leq$, but the set $\mathcal{T}(X)$ of all structured trees over X is not well-quasi-ordered by the relation $\leq_{tm}^{sl}$. By our assumption and Lemma 2.4.2, there exist a minimal (w.r.t. the number of vertices of the carrier of each of its terms) bad sequence on $\mathcal{T}(X)$. Let $(t_n)_{n\in\mathbb{N}}$ be such a sequence. For each $n \in \mathbb{N}$, we denote by r_n the root of the carrier of t_n.

Claim 3.2.19. *The sequence* $(\flat(t_n))_{n\in\mathbb{N}}$ *has finitely many one-node trees as its members.*

Proof of claim. Towards a contradiction we suppose that the claim does not hold, therefore the subsequence $(\flat(t_{k_n}))_{n\in\mathbb{N}}$ of all the one-node trees of $(\flat(t_n))_{n\in\mathbb{N}}$ is infinite. Then the sequence $(t_{k_n}(r_{k_n}))_{n\in\mathbb{N}}$ of the labels of the roots of its of those one-node trees is an infinite sequence on X and thus -by our assumption that X is well-quasi-ordered- the sequence $(t_{k_n}(r_{k_n}))_{n\in\mathbb{N}}$ is a good sequence on X and hence it has at least one good pair. Let $(t_{k_i}(r_{k_i}), t_{k_j}(r_{k_j}))$ be such a pair. Then (t_{k_i}, t_{k_j}) is a good pair of the sequence $(t_n)_{n\in\mathbb{N}}$, contradicting to our assumption that $(t_n)_{n\in\mathbb{N}}$ is a bad sequence. Thus the claim holds. $\square$

Let $(t_{l_n})_{n\in\mathbb{N}}$ be the infinite subsequence of $(t_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ the carrier of t_{l_n} has at least two vertices. Consider now the infinite sequence $(t_{l_n}(r_{l_n}))_{n\in\mathbb{N}}$ of the labels of the roots of the elements of $(t_{l_n})_{n\in\mathbb{N}}$. By our assumption that $\leq$ is a well-quasi-order on X and Theorem 2.1.7 it follows the existence of an infinite increasing subsequence of $(t_{l_n}(r_{l_n}))_{n\in\mathbb{N}}$. Let $(t_{l_{m_n}}(r_{l_{m_n}}))_{n\in\mathbb{N}}$ be such a sequence.

For each $n \in \mathbb{N}$, let $\{T_1, \ldots, T_{deg(r_{l_{m_n}})}\}$ be the set of the $deg(r_{l_{m_n}})$ connected components of the tree $\flat(t_{l_{m_n}}) \setminus r_{l_{m_n}}$. We consider each of these connected components as a structured tree with its root to be the vertex that neighboring with $r_{l_{m_n}}$ in $\flat(t_{l_{m_n}})$ and its structure to be the structured that is induced by $\flat(t_{l_{m_n}})$ and we set

$$A_{l_{m_n}} := \{t_{l_{m_n}}|_{V(T_i)} : 1 \leq i \leq deg(r_{l_{m_n}})\}.$$

Claim 3.2.20. *The set* $\mathcal{A} := \bigcup_{n\in\mathbb{N}} A_{l_{m_n}}$ *is well-quasi-ordered by the relation* $\leq_{tm}^{sl}$.

Proof of claim. To prove our claim it is sufficient to show that every sequence in $\mathcal{A}$ is a good sequence. Let $(t^k)_{k\in\mathbb{N}}$ be an arbitrary but fixed sequence in A. For each $k \in \mathbb{N}$, let $n = n(k)$ be such that $t^k \in A_{l_{m_n}}$. Let $n_0 = \min\{n(k)|k \in \mathbb{N}\}$ and k_0 be one arbitrary but fixed element of the set $\{k|n(k) = n_0\}$. Recall, that $(t_n)_{n\in\mathbb{N}}$ is a bad sequence, thus from the minimal choice of $t_{n(k)}$ and the fact that $V(t^k) \subsetneq V(t_{n_{k_0}})$, we derive that the sequence:

$$t_0, \ldots, t_{n(k_0)-1}, t^{k_0}, t^{k_0+1}, \ldots$$

is a good sequence and thus has at least one good pair. We will show that both elements of every good pair of this sequence belong at the sequence $t^{k_0}, t^{k_0+1}, \ldots$ and hence every such a pair is a good pair of the sequence $(t^k)_{k\in\mathbb{N}}$.

Let (t, t') be a good pair of the sequence $t_0, \ldots, t_{n(k_0)-1}, t^{k_0}, t^{k_0+1}, \ldots$. Since $(t_n)_{n\in\mathbb{N}}$ is a bad sequence, t cannot be among the first $n(k)$ elements $t_0, \ldots, t_{n(k_0)-1}$ of our sequence, because then: t' would be some t^i with $i \leq k_0$ and we would have:

$$t \leq_{tm}^{sl} t' = t^i \leq_{tm}^{sl} t_{n(i)}$$

Remembering the choice of k_0 we note that $n(k_0)-1 < n_{k_0} \leq n_i$, and thus $(t, t_{n(i)})$ is a good pair of $(t_n)_{n\in\mathbb{N}}$ contrary to the fact that $(t_n)_{n\in\mathbb{N}}$ is a bad sequence. Hence t indeed is not among the first $n(k)$ elements of the sequence $t_0, \ldots, t_{n(k_0)-1}, t^{k_0}, t^{k_0+1}, \ldots$ and thus (t, t') is a good pair of $(t^k)_{k\in\mathbb{N}}$. The sequence $(t^k)_{k\in\mathbb{N}}$ was an arbitrary sequence in A, and hence every sequence in $\mathcal{A}$ is a good sequence and the proof of our claim is complete. □

Note that since $\mathcal{A}$ is well-quasi-ordered by the relation $\leq_{tm}^{sl}$, Theorem 2.5.5 implies that the set $[\mathcal{A}]^{<\omega}$ is well-quasi-ordered by the extension of $\leq_{tm}^{sl}$. Hence the sequence $(A_n)_{n\in\mathbb{N}}$ of $[A]^{<\omega}$ has at least one good pair. Let (A_i, A_j) be such a pair and let $f : A_i \to A_j$ be the injection that witnesses the relation $A_i \sqsubseteq A_j$, then $t \leq_{tm}^{sl} f(t)$ for every $t \in A_i$ and thus for each $t \in A_i$ there is a monomorphism $\omega_t : t \to f(t)$ which witness the relation $t \leq_{tm}^{sl} f(t)$. We consider the function $\omega : V(t_i) \to V(t_j)$ with $\omega = (\bigcup_{t\in A_i} \omega_t) \cup (r_i, r_j)$. Notice that ω is a monomorphism which witness the relation $t_i \leq_{tm}^{sl} t_j$.
By our assumption that the set $\mathcal{T}(X)$ is not well-quasi-ordered by the relation $\leq_{tm}^{sl}$ we have derived the contradiction that the bad sequence $(t_n)_{n\in\mathbb{N}}$ has a good pair, the pair (t_i, t_j). Hence our assumption was false and the proof of Theorem 3.2.18 is complete. □

Corollary 3.2.21. The set $\mathcal{T}^\sharp$ is well-quasi-ordered by the relation $\leq_{tm}$ i.e by the topological minor relation.

Corollary 3.2.22. The Vazsonyi's Conjecture 3.1.6 holds.

Proof. It follows immediate from Proposition 3.2.2 and Corollary 3.2.21 □

3.2.2 A shorter proof of Vazsonyi's conjecture

Theorem 3.2.23. The set $\mathcal{T}^\sharp$ is well-quasi-ordered by the topological minor relation and hence by the minor relation.

Definition 3.2.24 (tree-order in rooted trees)**.** Given a rooted tree T, we define the binary relation tree-order $<$ on the set of its vertices as follows: Let $u, v \in V(T)$ then $u < v$ if and only if the unique path from the root to v passes through u.

We shall base the proof of Theorem 3.2.23 on the following notion of embedding between rooted trees.

Definition 3.2.25 ($\leq$ relation on $\mathcal{T}^\sharp$)**.** Let T, T' be two rooted trees and let r, r' be their roots respectively. We will write $T \leq T'$ if there exist an isomorphism ϕ from some subdivision of T to a subtree T'' of T', that preserves the tree-order on $V(T)$ i.e. if $x, y \in V(T)$ and $x < y$ in T, then $\phi(x) < \phi(y)$ in T'.

Observation 3.2.26. The above notion of embedding between rooted trees is stronger than the usual embedding that witnesses the topological minor relation between two graphs.

Proof of Theorem 3.2.23. From our last observation it follows that to prove the theorem it is sufficient to show that the set $\mathcal{T}^\sharp$ is well-quasi-ordered by the relation $\leq$ as we defined this relation on Definition 3.2.25.

Towards a contradiction let us suppose that the set $\mathcal{T}^\sharp$ is not well-quasi-ordered by the relation $\leq$. Thus by our assumption and Lemma 2.4.2, we can consider a minimal[8] bad sequence (w.r.t. $\leq$) in $\mathcal{T}^\sharp$. Let $(T_n)_{n\in\mathbb{N}}$ be such a sequence

For all $n \in \mathbb{N}$ we denote by r_n the root of the tree T_n and by A_n the set of the connected components of the graph $T_n \setminus r_n$, considering its one of those components as a rooted tree, with root the vertex that neighboring with r_n in T_n. We remark that the tree-order in each tree of the set A_n is the tree-order that induced by T_n. Our next goal is to prove the following claim:

Claim 3.2.27. *The set of rooted trees $A = \bigcup_{n\in\mathbb{N}} A_n$ is well-quasi-ordered by the relation $\leq$.*

Proof of claim. To prove our claim it is sufficient to show that every sequence in A is a good sequence. Let $(T^k)_{k\in\mathbb{N}}$ be an arbitrary but fixed sequence in A. For every $k \in \mathbb{N}$ we choose a natural number $n = n(k)$ such that $T^k \in A_n$. Let $n_0 = \min\{n(k)|k \in \mathbb{N}\}$ and k_0 be one arbitrary but fixed element of the set $\{k|n(k) = n_0\}$. Then from the minimal choice of $T_{n(k)}$ and the fact that $T^k \subsetneq T_{n_{k_0}}$, we derive that the sequence

$$T_0, \dots, T_{n(k_0)-1}, T^{k_0}, T^{k_0+1}, \dots$$

is a good sequence. We will show that both elements of every good pair of this sequence belong at the sequence $T^{k_0}, T^{k_0+1}, \dots$ and hence every such a pair is a good pair of the sequence $(T^k)_{k\in\mathbb{N}}$. Let (T, T') be a good pair of the sequence $T_0, \dots, T_{n(k_0)-1}, T^{k_0}, T^{k_0+1}, \dots$. Since $(T_n)_{n\in\mathbb{N}}$ is a bad sequence, T cannot be among the first $n(k)$ elements $T_0, \dots, T_{n(k_0)-1}$ of our sequence, because then: T' would be some T^i with $i \leq k_0$ and we would have:

$$T \leq T' = T^i \leq T_{n(i)}$$

Remembering the choice of k_0 we note that $n(k_0) - 1 < n_{k_0} \leq n_i$, and thus $(T, T_{n(i)})$ is a good pair of $(T_n)_{n\in\mathbb{N}}$ contrary to the fact that $(T_n)_{n\in\mathbb{N}}$ is a bad sequence. Hence T indeed is not among the first $n(k)$ elements of the sequence $T_0, \dots, T_{n(k_0)-1}, T^{k_0}, T^{k_0+1}, \dots$ and thus (T, T') is a good pair of $(T^k)_{k\in\mathbb{N}}$. The sequence $(T^k)_{k\in\mathbb{N}}$ was an arbitrary but fixed sequence in A, and hence every sequence in A is a good sequence and the proof of our claim is complete. ☐

Note that since A is well-quasi-ordered by the relation $\leq$, Theorem 2.5.5 implies that the set $[A]^{<\omega}$ is well-quasi-ordered by the expansion of $\leq$ on it. Hence the sequence $(A_n)_{n\in\mathbb{N}}$ of $[A]^{<\omega}$ has at least one good pair, let (A_i, A_j) be such a pair. Let $f : A_i \to A_j$ be the injection that witnesses the relation $A_i \leq A_j$, then $T \leq f(T)$ for every $T \in A_i$. For each $T \in A_i$ there is an embedding ϕ_T that witnesses the relation $T \leq f(T)$. We consider the function $\phi : V(T_i) \to V(T_j)$ with $\phi = (\bigcup_{T\in A_i} \phi_T) \cup (r_i, r_j)$. Now the function ϕ is an embedding that witnesses the relation $T_i \leq T_j$. By our assumption that the set $\mathcal{T}^\sharp$ is not well-quasi-ordered by the relation $\leq$ we have derive the contradiction that the bad sequence $(T_n)_{n\in\mathbb{N}}$ has a good pair, the pair (T_i, T_j). Hence our assumption was false and the proof of Theorem 3.2.23 is complete. ☐

3.3 Symmetric submodular functions and branch-decompositions

In this section we consider symmetric submodular functions, we introduce the concept of their branch-decompositions, we define the branch-width of a symmetric submodular function and we

[8]The minimality is relevant to the number of vertices of each element of the sequence.

prove some results which will be used in the proofs of some of the main results of this chapter such as the well-quasi-ordering of graphs of bounded branch-width by the minor relation.

A set function is said to be submodular if it has the property that the difference in the incremental value of the function that a single element makes when added to an input set decreases as the size of the input set increases.

The symmetric submodular functions that we will consider throughout this chapter are the connectivity functions of graphs. The branch-decompositions and branch-width of a given graph are the branch-decompositions and branch-width of its connectivity function. The graph invariant of branch-width is a measure of "global connectivity" of the graph, broadly speaking, a graph has small branch-width if it can be decomposed across non-crossing separations into small pieces. Branch-width was first defined by Roberson and Seymour in [111], a survey of results, open problems and bibliography considering the branch-width of graphs can be found in [49].

Definition 3.3.1 (submodular function)**.** Given a finite set S and a function λ defined on $\mathcal{P}(S)$, the function λ is called *submodular* if the following condition holds:

$$(\forall A, B \in \mathcal{P}(S))[\lambda(A) + \lambda(B) \geq \lambda(A \cap B) + \lambda(A \cup B)]$$

The set S is called the ground set of λ.

Definition 3.3.2 (symmetric submodular function)**.** Given a finite set S and a submodular function λ whose ground set is S, the function λ is called *symmetric* if the following condition holds:

$$(\forall A \in \mathcal{P}(S))[\lambda(A) = \lambda(S \setminus A)]$$

Definition 3.3.3 (partial branch-decomposition of a symmetric submodular function)**.** Given a finite set S and a symmetric submodular function λ that has as ground set the set S, a *partial branch-decomposition* of λ is a pair $B = (T, \tau)$, where T is a cubic tree T and $\tau : S \to L(T)$ is an onto function, mapping the elements of S to the leaves of T. A leaf $l \in L(T)$ will be said to be *loaded* if $|\tau^{-1}(l)| > 1$. An edge of T is *loaded* if it is incident to some loaded leaf.

Definition 3.3.4 (valency of a partial branch-decomposition)**.** Let B be a partial branch-decomposition of a symmetric submodular function. We define as the *valency* of B the number $|L(T)|$.

Definition 3.3.5 (incomplete and complete partial branch-decomposition)**.** Let S be a finite set, let also λ a symmetric submodular function that has as ground set the set S and $B = (T, \tau)$ be a partial branch-decomposition of λ. We will call B *incomplete* if $|S| > |L(T)|$ and *complete* if $|S| = |L(T)|$. In this case, τ is a bijection from $E(G)$ to $L(T)$.

Definition 3.3.6 (branch-decomposition of a symmetric submodular function)**.** Given a symmetric submodular function λ, a *branch-decomposition* of λ, is a complete partial branch-decomposition of λ.

Notation 3.3.7 ($B = (T, \tau) \equiv T$)**.** Let $B = (T, \tau)$, be a branch-decomposition of a symmetric submodular function λ. When our purposes do not impose us to refer to the function that corresponds to a branch-decomposition we identify the reference to the branch-decomposition with the reference to its correspondent cubic tree, i.e. we refer to the cubic tree T as the branch-decomposition of λ. In particular, this is the case for the proofs of Theorems 3.3.21, 3.4.2.

Comment 3.3.8 (unlabeled leaves)**.** *In the proofs of Theorems 3.3.21, 3.4.2 we allow -by violating Definition 3.3.6- for convenience and for technical reasons respectively, a branch-decomposition to have some leaves that do not correspond to elements of the ground set of the symmetric submodular function. We call such leaves* unlabeled *and remark that branch-decompositions with unlabeled leaves are easily turned onto branch-decompositions with the same width but no unlabeled leaves: just delete the unlabeled leaves and suppressing vertices of degree 2 until the tree is cubic again.*

Notation 3.3.9. Let S be a finite set and λ submodular function on $\mathcal{P}(S)$. Given two disjoint subsets A and B of S we denote by $\lambda(A, B)$ the value

$$\min\{\lambda(X)|(X \subseteq S) \wedge (A \subseteq X) \wedge (X \cap B = \emptyset)\}$$

Clearly if λ is a symmetric submodular function, then:

$$(\forall A, B \in \mathcal{P}(S))[A \cap B = \emptyset \Rightarrow \lambda(A, B) = \lambda(B, A)].$$

Definition 3.3.10. Let S be a finite set, λ a symmetric submodular function whose ground set is S, and let $B = (T, \tau)$ be branch-decompositions of λ.

- Given a subtree T' of T, let $A = \tau^{-1}(L(T) \cap L(T'))$, we will say that A is *displayed* by T'.
- A subset of S is *displayed by an edge* e of T if it is displayed by one of the two connected components of $T \setminus e$.

Observation 3.3.11. Whenever two sets are displayed by edges in a branch-decomposition are either disjoint, or their union is equal with the ground set of the symmetric submodular function or are comparable by inclusion.

Definition 3.3.12 (width of an edge of a branch-decomposition)**.** Given a finite set S, a symmetric submodular function λ whose ground set is S, a branch-decomposition $B = (T, \tau)$ of λ and an edge e in T, we define the *width* of e to be the value that the function λ takes at the one of the two sets displayed by e, note that due to the symmetry of λ this two values are equal. We denote the width of e by $\lambda(e)$.

Definition 3.3.13 (width of a branch-decomposition of a symmetric submodular function)**.** The width of a branch-decomposition is the maximum of the widths of its edges.

Definition 3.3.14 (branch-width of a symmetric submodular function)**.** Given a symmetric submodular function the branch-width of that function is defined to be the minimum of the widths of all its branch-decompositions.

3.3.1 A Menger-like property of branch-width

The notion of tree-width is a cornerstone of Robertson and Seymour's Graph Minors series [104]. Informally the graph invariant of tree-width is a measure of the "tree-likeness" of a graph. Although we will not make use of the notions of tree-decompositions and tree-width at any of our proofs, we define these notions formally below, because in the course of this chapter we refer a couple of times to those and in order to be able to express some results considering tree-decompositions and tree-width which are closely related with the main result of this section, which is Theorem 3.3.21, and it will be used in the proof of the well-quasi-ordering of graphs of bounded branch-width by the minor relation.

Definition 3.3.15 (tree-decomposition, width of a tree-decomposition, tree-width of a graph). Given a graph G a *tree-decomposition* of G is an ordered pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X} = \{X_t \subseteq V(G) | t \in V(T)\}$ is a collections of subsets of the vertex set of G, called *bags* with the following properties:

(i) $\bigcup_{t \in V(T)} X_t = V(G)$;

(ii) $(\forall e \in E(G))(\exists t \in V(T))[e \subseteq X_t]$, that is every edge belongs to some bag; and

(iii) for every $u \in V(G)$, the set of vertices $\{t \in V(T) | u \in X_t\}$ induce a connected subgraph of T.

The *width* of a tree-decomposition $(T, \mathcal{X})$ is $\max\{|X_t| | t \in V(T)\} - 1$, the *tree-width* of a graph G, denoted by $\mathbf{tw}(G)$, is defined as the minimum width of a tree decomposition of G.

Roberson and Seymour in [109] proved Theorem 3.4.1 which states that any set of graphs of bounded tree-width is well-quasi-ordered by the minor relation. Due to the fact (Theorem 3.3.33) that branch-width and tree-width are within a constant factor, the statement of Theorem 3.4.1 is equivalent with the statement that any set of graphs of bounded branch-width is well-quasi-ordered by the minor relation. In order to derive the proof of Theorem 3.4.1 Roberson and Seymour proved Theorem 3.3.17 which states the existence of tree-decompositions of "small" width which satisfies a certain vertex-connectivity condition.

Definition 3.3.16 (linked tree-decomposition). A tree decomposition $(T, \mathcal{X})$ will be said to be *linked* if for all $k \in \mathbb{N}$ and every $t, t' \in V(T)$, either G contains k disjoint $(X_t, X_{t'})$-paths or there is a vertex $c \in V(T)$ in the unique path between t and t' in T, such that $|X_c| < k$.

Theorem 3.3.17 (Robertson and Seymour [108]). Every graph G admits a linked tree-decomposition of width less than $3.2^{\mathbf{tw}(G)}$.

The exponential upper bound $3 \cdot 2^{\mathbf{tw}(G)}$ for the width of the tree-decomposition the existence of which is guaranteed by Theorem 3.3.17, was improved by Thomas [121] to its optimal value.

Theorem 3.3.18 (Thomas [121]). Every graph G admits a linked tree-decomposition of width $\mathbf{tw}(G)$.

In fact Thomas showed a stronger result -which we will not state here-, which states that every graph G has a *lean tree-decomposition* of width $\mathbf{tw}(G)$. Later Kříž and Thomas [80] extended Theorem 3.3.18 for infinite graphs, and Thomas [120] used it to prove Theorem 3.6.27 which states that given any finite planar graph H the set of all graphs (finite and infinite) with no minor isomorphic to H is well-quasi-ordered (actually the better-quasi ordered) by the minor relation. There is a short proof of Theorem 3.3.18 by Bellenbaum and Diestel [10].

Similar results have been proved for several different width-parameters such as tree-cut width [57], θ-tree-width [18, 56], path-width [85], directed path-width [73], DAG-width [75], rank-width [98], linear- rank-width [67], profile-width and block-width [40], matroid tree-width [8, 40, 53], matroid branch-width [53].

The main result of this section is Theorem 3.3.21 which is the analog of Thomas' Theorem 3.3.18 for branch-decompositions. We first need to define the notion of linked branch-decomposition.

Definition 3.3.19 (linked edges)**.** Let S be a finite set, λ a symmetric submodular function whose ground set is S and let $B = (T, \tau)$ be branch-decompositions of λ. Let e_1, e_2 be two edges of T, let E_1 be the set displayed by the component of $T \setminus e_1$ that does not contain e_2 and let E_2 be the set displayed by the component of $T \setminus e_2$ that does not contain e_1. Let P be the shortest path in T that contains both e_1 and e_2, then each edge on P displays a subset of S that contains E_1 and is disjoint from E_2, thus the widths of the edges of P are upper bounds for the value $\lambda(E_1, E_2)$. We call e_1, e_2 *linked* if $\lambda(E_1, E_2)$ is equal to the minimum width of an edge on P. It's immediate from the definition that every edge is linked to itself.

Definition 3.3.20 (linked branch-decomposition)**.** A branch-decomposition of a symmetric submodular function will be said to be *linked* if and only if all its edge pairs are linked.

Theorem 3.3.21. For every positive integer n, if λ is an integer-valued symmetric submodular function with branch-width n, then λ has a linked branch-decomposition of width n.

Proof. Let $n \geq 1$ be an arbitrary but fixed positive integer and λ be an arbitrary but fixed integer-valued symmetric submodular function, with ground set, the set S and branch-width n. As a first step to our proof we will equip the set $\mathcal{D}$ of all branch-decompositions of λ with a strict partial order $<$, then the core of our proof lies in the proof of Lemma 3.3.26, which states that every minimal (w.r.t $<$) element of $\mathcal{D}$ is a linked branch-decomposition of λ and since by Claim 3.3.25 every such an element is a branch decomposition of width n of λ we will obtain the desired result.

For each branch-decomposition T of λ and each natural number k, we define T_k to be the forest which is induced by the edges of T with width at least k. Note that subgraphs induced by edges have no isolated vertices. For a graph H we denote by $e(H)$ the number of edges of H and by $c(H)$ the number of connected components of H.

Let $T, R \in \mathcal{D}$, we write $T < R$ if there exists a number k such that:

(i) $\big(e(T_k) < e(R_k)\big) \vee \big((e(T_k) = e(R_k)) \wedge (c(T_k) > c(R_k))\big)$

(ii) $(\forall n \in \mathbb{N})[n > k \Rightarrow \big((e(T_n) = e(R_n)) \wedge (c(T_n) = c(R_n))\big)]$

Comment 3.3.22. *Recall Definition 3.3.13 and notice that by comparing two branch-decompositions with only criterion their width, we are only looking at the width of their wider edges and we ignore how many edges have this width and what happens with the width of their other edges. By the definition of the strict partial order $<$ on $\mathcal{D}$, we intend to distinguish branch-decompositions of the same width, when the one is in some sense better than the other. The underlying criterion to choose among two branch-decompositions of the same width which motivates the particular definition of $<$ is the following:*

Between two branch-decompositions of different width of course we prefer the one that has the minimum width. Among two branch-decompositions of the same width the one, say T, may be more preferable than the other, say R, if there exist a natural number k such that:

(i) the number of the edges of width k of T is less than the corresponding number of R or it is equal, but T has more edges of width less than k than R, and

(ii) for every natural number $n > k$ the number of edges of width n is equal to both T and R and so does the number of edges of width less than n.

Claim 3.3.23. *The set $\mathcal{D}$ is strictly partially ordered by the relation $<$ that we defined above.*

Proof of claim. It is sufficient to show that the relation $<$ is irreflexive, transitive and antisymmetric on $\mathcal{D}$. Let $T, R, H \in \mathcal{D}$ be three arbitrary but fixed branch-decompositions of λ such that $T < R, R < H$. Then:

- Irreflexivity: $T \not< T$, because for all natural numbers the first condition does not hold.
- Transitivity: Let k_1, k_2 be the natural numbers that witness the relations $T < R$ and $R < H$ respectively, then the natural number $k = \max\{k_1, k_2\}$ witnesses that $T < H$.
- Antisymmetry: The relation $R < T$ -through the transitivity of $<$- would imply $T < T$ which contradicts the irreflexivity of $<$, thus $R \not< T$.

Since T, R, H where arbitrary the claim follows. ☐

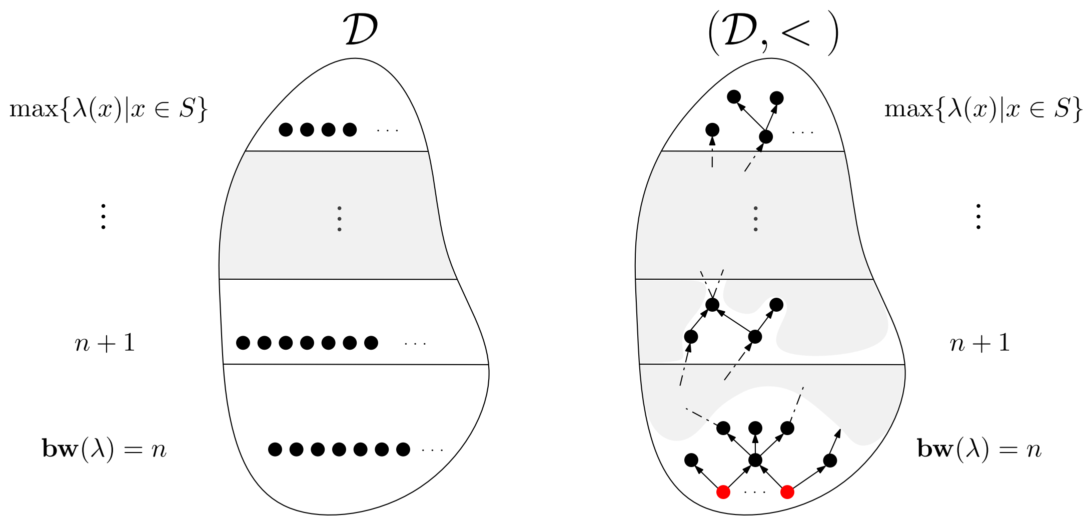

Figure 3.3.1: The strict partial order $<$ on the set $\mathcal{D}$ of all branch-decompositions of λ distinguish branch-decompositions of the same width in the way that is illustrated on Comment 3.3.22. Each minimal (w.r.t $<$) element of $\mathcal{D}$ is a linked branch-decomposition of λ.

Claim 3.3.24. *The set $\mathcal{D}$ has at least one minimal (w.r.t $<$) element.*

Proof of claim. That's true for every strictly partially ordered set, it follows from the fact that every partial order can be seen as a DAG (directed acyclic graph), and since every DAG has at least one vertex with in-degree 0 the claim follows because every such vertex corresponds to a minimal element of the partial order. ☐

Claim 3.3.25. *Every minimal (w.r.t $<$) element of $\mathcal{D}$ has width n.*

Proof of claim. Let T be a minimal element of D. Towards a contradiction suppose that the width of T is $k > n$. Let $l = \max\{m \in \mathbb{N} | (m > n) \wedge (m \leq k)\}$ and let R be a branch-decomposition of width n of λ, then l witnesses that $R < T$ contradicting to the minimality of T. ☐

Lemma 3.3.26. Every minimal (w.r.t $<$) element of $\mathcal{D}$ is a linked branch-decomposition of λ.

Proof of Lemma 3.3.26. Let T be a minimal element of $\mathcal{D}$. Towards a contradiction we suppose that T is not linked. Let f, g be to edges in T witnesses that that T is not linked i.e. f, g is a pair of unlinked edges. Clearly $f \neq g$, since every edge is linked with itself. Let F be the set displayed by the connected component of $T \setminus f$ that does not contain g and G be the set displayed by the connected component of $T \setminus g$ that does not contain f. Let x, y be the end vertices of f, g respectively such that the unique $xy-$path in T does not contain neither f nor g.

Given $X, Y \subseteq S$, we say that X *splits* Y if $X \cap Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$. Note that splitting is not a symmetric binary relation on $\mathcal{P}(S)$ i.e. X splitting Y does not implies Y splitting X.

We choose a subset A of $S \setminus G$ such that $F \subseteq A$, $\lambda(A) = \lambda(F, G)$, and A splits as few subsets of S displaying by edges in T as possible. Note, that such a choice can be done since the set $\{B | (B \subseteq S) \wedge (B \cap G = \emptyset) \wedge (F \subseteq B) \wedge (\lambda(B) = \lambda(F, G))\}$ is non-empty.

We now proceed to the construction of a branch-decomposition $\hat{T}$ of λ (see Figure 3.3.2) from which we will derive the desired contradiction. Let T^+ be a copy of the connected component of $T \setminus g$ that contains the edge f and T^- be a copy of the connected component of $T \setminus f$ that contains the edge g.

Then the tree $\hat{T}$ consists in T^+ and T^- connected with a new edge α incident to the copies of x and y in $T^-$and T^+ respectively, $\hat{T}$ is clearly cubic. We turn $\hat{T}$ into a branch-decomposition of λ as follows: Each element s of S -which is a leaf of T - is identified with its copy in T^+ if $s \in A$ and with its copy in T^- otherwise. Notice, that from its construction $\hat{T}$ contains twice the connected component of $T \setminus \{f, g\}$ that contains x and y.

Claim 3.3.27. *Let e be an edge in T and $\hat{e}$ one of its copies in $\hat{T}$. Then $\lambda(\hat{e}) \leq \lambda(e)$, and the equality holds only if e has at most one copy in $\hat{T}_{\lambda(A)+1}$.*

Proof of Claim 3.3.27. In order to prove this, by symmetry, there is no loss of generality if we suppose that $\hat{e}$ lies in T^+. Let W be the set displayed by the component of $T \setminus e$ that does not contain the vertex y. Then for the width of the edges $e, \hat{e}$ we have $\lambda(e) = \lambda(W)$ and $\lambda(\hat{e}) = \lambda(A \cap W)$ respectively. From the submodularity of λ we have: $\lambda(A \cap W) + \lambda(A \cup W) \leq \lambda(A) + \lambda(W)$, and thus we have:

$$\lambda(\hat{e}) + \lambda(A \cup W) \leq \lambda(A) + \lambda(e) = \lambda(F, G) + \lambda(e) \leq \lambda(A \cup W) + \lambda(e).$$

It follows that $\lambda(\hat{e}) \leq \lambda(e)$, and the equality holds only if $\lambda(W \cup A) = \lambda(A)$.

To complete the proof of Claim 3.3.27 it is sufficient to show that whenever the equality $\lambda(\hat{e}) = \lambda(e)$ holds the edge e has at most one copy in $\hat{T}_{\lambda(A)+1}$, and hence from the above it is sufficient to show that whenever $\lambda(W \cup A) = \lambda(A)$, the edge e has at most one copy in $\hat{T}_{\lambda(A)+1}$. For that, let $\lambda(W \cup A) = \lambda(A)$, then $\lambda(W \cup A) = \lambda(F, G)$.

Claim 3.3.28. *The set A does not split W.*

Proof of Claim 3.3.28. Towards a contradiction we suppose that A splits W. Note that by the choice of the set A, it follows that the set $W \cup A$ splits at least as many subsets of S displayed by edges in T as A does. Since $W \setminus (W \cup A) = \emptyset$ it follows that $W \cup A$ does not split W and thus there exists a set Y displayed by an edge of T, such that the set $W \cup A$ splits Y but A does not. Since $W \cup A$ splits Y we have $Y \setminus (W \cup A) \neq \emptyset$ and thus $Y \setminus A \neq \emptyset$, since A does not split Y

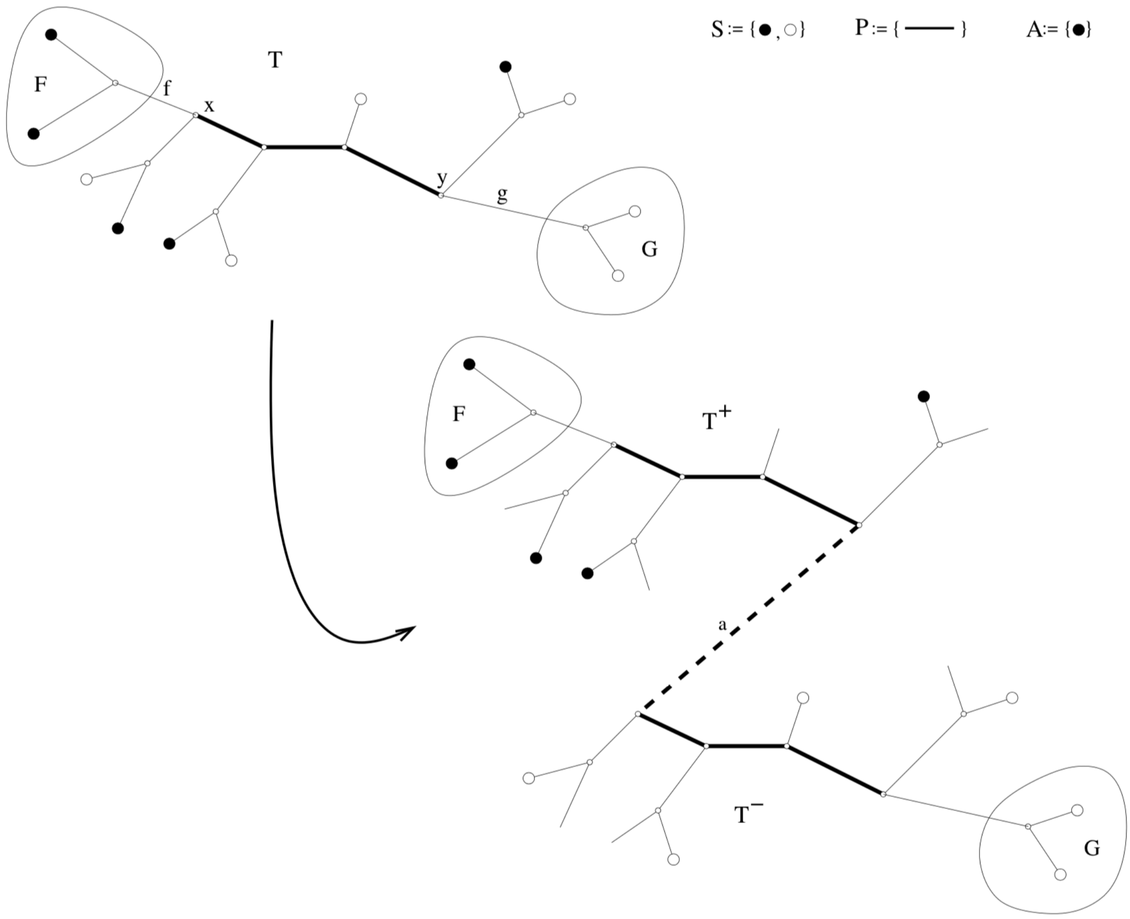

Figure 3.3.2: This figure is taken from [55] and illustrates the proof of Lemma 3.3.26.

this implies that $A \cap Y = \emptyset$ and W splits Y. As A splits W the fact that $A \cap Y = \emptyset$ implies that $W \setminus Y \neq \emptyset$. As Y and W are both displayed by edges of T and as W splits Y Observation 3.3.11 implies that $Y \cup W = S$, hence $A \subseteq W$ and thus $F \subseteq W$. Moreover as $\hat{e}$ lies in T^+, the choice of W is such that W lies in $S \setminus G$. Hence, as the edges f, g are not linked in T, we have $\lambda(F, G) < \lambda(W) = \lambda(e) = \lambda(\hat{e}) = \lambda(W \cap A) = \lambda(A)$. We have thus derived the contradiction that $\lambda(A) > \lambda(F, G)$. Hence our assumption was false and the proof of the claim is complete. □

Since A does not split W, it follows that either $A \cap W = \emptyset$, or $W \setminus A = \emptyset$. Note that combining the symmetry and the submodularity of λ we take:

$$(\forall B \subseteq S)[2\lambda(B) = \lambda(B) + \lambda(S \setminus B) \geq \lambda(\emptyset) + \lambda(S) = 2\lambda(\emptyset)].$$

It follows, that $(\forall B \subseteq S)[\lambda(B) \geq \lambda(\emptyset)]$. We now distinguish the following two cases:

Case 1: $A \cap W = \emptyset$. In this case $\lambda(\hat{e}) = \lambda(A \cap W) = \lambda(\emptyset) \leq \lambda(A) < \lambda(A) + 1$. Thus, in this case $\hat{e} \notin \hat{T}_{\lambda(A)+1}$. So if e has a second copy in $\hat{T}$, only this copy is possible to belong in $\hat{T}_{\lambda(A)+1}$. And hence in this case e has at most one copy in $\hat{T}_{\lambda(A)+1}$.

Case 2: $W \setminus A = \emptyset$. If e has only one copy in $\hat{T}$ we are done. So let us suppose that e has a second copy in $\hat{T}$, say e^*, then this copy will lie in T^-.

- If $e \in P$ then $\lambda(e^*) = \lambda(W \cup A) = \lambda(A) < \lambda(A) + 1$
- If $e \notin P$ then $\lambda(e^*) = \lambda(W \setminus A) = \lambda(\emptyset) \leq \lambda(A) < \lambda(A) + 1$

In any case $\lambda(e^*) < \lambda(A) + 1$ and hence $\lambda(e^*) \notin \hat{T}_{\lambda(A)+1}$. Thus, at most one copy of e lies in $\hat{T}_{\lambda(A)+1}$, and that completes the proof of Claim 3.3.27. □

We now return to the proof of Lemma 3.3.26. Let us denote by B the set:

$$\{n \in \mathbb{N} \,|(n > \lambda(A)) \wedge \big((\forall k > n)[e(T_k) = e(\hat{T}_k)]\big)\}.$$

Note that if we set $l = \max\{\lambda(e)|(e \in T) \vee (e \in \hat{T})\}$ then for every integer k greater than l both T_k and $\hat{T}_k$ have zero edges and hence B is nonempty, we can thus choose the minimum element of B. Let $p = \min\{n|n \in B\}$,

- From Claim 3.3.27 it follows that for each $k \geq p$ each edge of T_k has at most one copy in $\hat{T}_k$,
- Moreover since A is displayed by α in $\hat{T}$ we have $\lambda(\alpha) = \lambda(A)$ and hence $\alpha \notin \hat{T}_k$ for $k > \lambda(A)$.

So for every $k \geq p$ we have $e(T_k) \geq e(\hat{T}_k)$, with $c(T_k) \leq c(\hat{T}_k)$ whenever $e(T_k) = e(\hat{T}_k)$. However from the minimal choice of T we can't have $\hat{T} < T$ so in fact:

$$(\forall k \geq p)[(e(T_k) = e(\hat{T}_k)) \wedge (c(T_k) = c(\hat{T}_k))].$$

Thus also T_p and $\hat{T}_p$ have the same number of edges, which by definition of p implies that $p = \lambda(A) + 1$. Moreover, as $c(T_{\lambda(A)+1}) = c(\hat{T}_{\lambda(A)+1})$ each component of $T_{\lambda(A)+1}$ is copied entirely and as one in $\hat{T}_{\lambda(A)+1}$. In particular this holds also for the component $P \cup \{f, g\}$, which lies entirely in $T_{\lambda(A)+1}$. This is absurd, f has a copy only in T^+, g has a copy only in T^-, and α is not in $T_{\lambda(A)+1}$. We have thus derived a contradiction by assuming that T is not linked, hence T is indeed linked and the proof of Lemma 3.3.26 is complete. □

As n was an arbitrary positive integer and λ was an arbitrary integer-valued symmetric submodular function of branch-width n, the proof of Theorem 3.3.21 is complete. □

3.3.2 Branch-decompositions & branch-width of graphs

The symmetric submodular functions that we will consider in the course of this chapter are the connectivity functions of graphs.

Definition 3.3.29 (connectivity function of a graph)**.** Given a graph $G = (V, E)$, for $A \subseteq E$ we denote by $\Gamma_G(A)$ the set of vertices that are incident with an edge in A and also with an edge in $E \setminus A$. We define the *connectivity function* of G to be the function $\gamma_G : \mathcal{P}(E) \to \mathbb{N}$, with $\gamma_G(A) := |\Gamma_G(A)|$.

Observation 3.3.30. Given a graph $G = (V, E)$, it is immediate from Definition 3.3.29 to see that:

- $(\forall A \subseteq E)[\gamma_G(A) = \gamma_G(E \setminus A)]$
- $(\forall A, B \subseteq E)[(\gamma_G(A) \geq \gamma_G(A \cap B)) \wedge (\gamma_G(A) + \gamma_G(B) \geq \gamma_G(A \cup B))]$, hence $(\forall A, B \subseteq E)[\gamma_G(A) + \gamma_G(B) \geq \gamma_G(A \cap B) + \gamma_G(A \cup B)]$.

Thus, the connectivity function of a graph, is a symmetric submodular function.

Definition 3.3.31 ((partial) branch-decomposition and branch-width of a graph)**.** Given a graph G, a (partial) branch-decomposition of G is a (partial) branch-decomposition of its connectivity function. The branch-width of G is defined to be the branch-width of its connectivity function.

Notation 3.3.32 (branch-width of a graph)**.** Given a graph G, we denote the branch-width of G by $\mathbf{bw}(G)$.

Theorem 3.3.33 (Robertson and Seymour [111])**.** Let G be a graph, then $\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \lfloor 3/2\, \mathbf{bw}(G) \rfloor$.

Comment 3.3.34. *It immediate from Definition 3.3.31, that the branch-width of a graph is equal with the maximum branch-width of its connected components and that if two graphs, say G, H are isomorphic then $\mathbf{bw}(G) = \mathbf{bw}(H)$.*

Definition 3.3.35 (middle set of an edge of a partial branch-decomposition)**.** Let G be a graph, $B = (T, \tau)$ be a (partial) branch-decomposition of G, e be an edge of T and A a set displayed be e in T. We define as the *middle set* of e the set $\Gamma_G(A)$. We will denote the middle set of e by $\mathbf{mid}(e)$. Note that $|\mathbf{mid}(e)| = |\Gamma_G(A)| = \gamma_G(A)$.

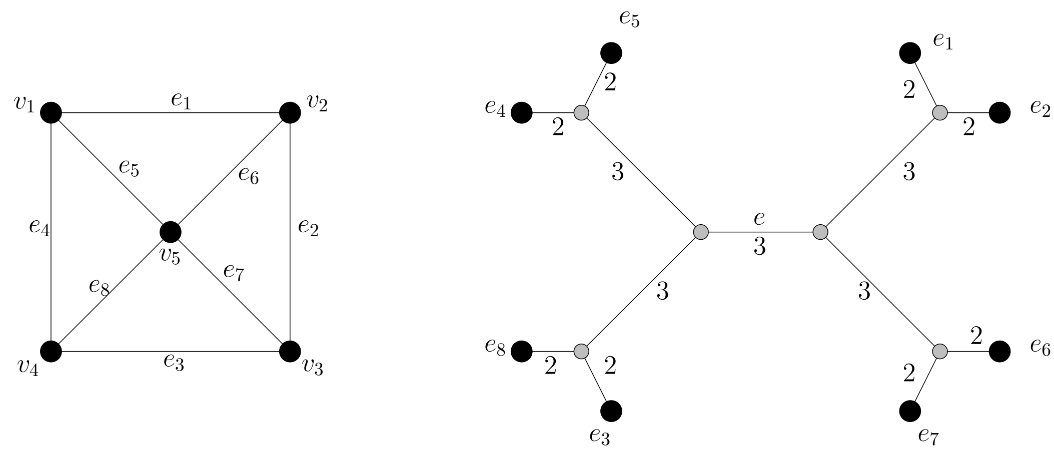

Figure 3.3.3: On the left hand side it is illustrated a graph and on the right hand side a branch-decomposition of this graph. $\mathbf{mid}(e) = \{v_1, v_3, v_5\}$.

Observation 3.3.36. Given any graph G for any two sets of edges of G, say E_1, E_2, the value $\gamma_G(E_1, E_2)$ equals to the size of the minimum $(\Gamma(E_1), \Gamma(E_2))$-separator and thus by by Theorem 1.2.73 with the maximum number of mutually internally vertex-disjoint $(\Gamma(E_1), \Gamma(E_2))$-paths.

The following theorem is an immediate corollary of Theorem 3.3.21.

Theorem 3.3.37. Let n be a positive integer and G be a graph such that $\mathbf{bw}(G) = n$, then G has a linked branch-decomposition of width n.

Proposition 3.3.38. Let G, H be graphs. If H is isomorphic to a minor of G, then $\mathbf{bw}(H) \leq \mathbf{bw}(G)$.

Proof. We may assume that $|E(H)| \geq 2$, since otherwise $\mathbf{bw}(H) = 0$ and the is nothing to prove. Let $B = (T, \tau)$ be a branch-decomposition of G with width $\mathbf{bw}(G)$. Let S be a minimal subtree of T such that $(\forall e \in E(H))[\tau^{-1}(e) \in L(S) \cap L(T)]$. Let T' be the cubic tree obtained from S by suppresing all vertices of degree 2 on it (that is, for any vertex of degree 2 we delete this vertex and its incident edges and we add a new edge joining its neighbors and we continue this process until no such vertices remain). Let $\tau^{'}$ be the restriction of T to the set $L(T^{'}) = L(T) \cap L(S) \subseteq L(T)$, then $B' = (T^{'}, \tau^{'})$ is a branch-decomposition of H and its width is $\leq \mathbf{bw}(G)$, thus $\mathbf{bw}(H) \leq \mathbf{bw}(G)$ and the proof of the proposition is complete. □

The following is an immediate corollary of [50, Lemma 3.1].

Theorem 3.3.39. The branch-width of every graph is equal to the maximum branch-width of its blocks.

3.4 Graphs with bounded branch-width

Robertson and Seymour on [109] proved the following theorem:

Theorem 3.4.1 (Robertson and Seymour [109, Theorem 1.5])**.** For each positive integer n the set of all graphs with tree-width at most n is well-quasi-ordered by the minor relation.

Theorem 3.4.1 is a central result of the Graph Minors series, for example it is one of the two basic ingredients for the proof (presented in Section 3.6) of the Robertson ans Seymour's theorem (Theorem 3.1.11) in the case that the graph G_1 is planar. Another result in which Theorem 3.4.1 plays a crucial role, is the "Kuratowski's theorem for General Surfaces" (Theorem 3.1.5) whose proof is presented in Section 3.7.

Recall the close relationship of the graph parameters branch-width and tree-width (Theorem 3.3.33) that implies the equivalent of Theorem 3.4.1 with the following theorem:

Theorem 3.4.2. For each positive integer n the set of all graphs with branch-width at most n is well-quasi-ordered by the minor relation.

In this section we give a proof of Theorem 3.4.2. The reason why we state Theorem 3.4.1 in terms of branch-width is because working with branch-width makes the proof much simpler and shorter. The proof of Theorem 3.4.2 is due to Geelen, Gerards, and Whittle [53]. In a lot points of the proof we follow the presentation of Richter [102].

Another formulation of Theorem 3.4.2 wehich is useful for applications is the following.

Theorem 3.4.3. Let $\mathcal{G}$ be a set of graphs of bounded branch-width, then $\mathcal{G}$ is well-quasi-ordered by the minor relation.

The proof of Theorem 3.4.2 will run roughly as follows: We suppose towards a contradiction that the theorem does not hold. Let n be an integer which witnesses our assumption, that is, the set of graphs with branch-width $\leq n$ -denoted by $\mathcal{G}_n$- is not well-quasi-ordered by the minor relation. We will work in the forest which consists of the branch-decompositions of all the graphs of $\mathcal{G}_n$. In subsection 3.3.1 we prove "Lemma on Trees" (Lemma 3.4.9) which gives us some information about the structure that this forest must have (actually in subsection 3.3.2 we prove a corollary of the "Lemma on Trees" (Lemma 3.4.12) which is exactly what we will use). We will prove that this structure does not occur in our forest, deriving by this way the desired contradiction.

3.4.1 Lemma on trees

"Lemma on Trees" was first proved on [109], the proof that we presented here is from [53, 55].

Definition 3.4.4 (Rooted forest)**.** A *rooted forest* is a collection of countable many vertex disjoint rooted trees. Its vertices with in-degree 0 are called *roots* and those with out-degree 0 are called leaves. The edges that are incident to a root are called *root edges* and those that are incident to a leaf are called *leaf edges*.

Notation 3.4.5. Given a rooted forest F and a set of edges S in F, we denote by $u_F(S)$ the set of those edges in F whose tail is a head of an edge in S.

Definition 3.4.6 ($n-$edge labeling of a graph)**.** Given a graph G an *$n-$edge labeling* of a graph is a map from the set of the edges of G to the set $\{1, \ldots, n\}$.

Definition 3.4.7 ($\lambda-$ linked)**.** Given a rooted forest F, a function λ that is $n-$edge labeling of F and two edges e, f in F we say that *e is $\lambda-$linked to f* if F contains a directed path P starting with e and ending with f such that $\lambda(g) \geq \lambda(e) = \lambda(f)$ for each edge g on P.

Observation 3.4.8. Given a rooted forest F and a function λ that is $n-$edge labeling of F, from the above definition it follows that every edge is $\lambda-$linked to itself.

Lemma 3.4.9 (Lemma on Trees)**.** Let F be a rooted forest with an $n-$edge labeling λ. Moreover, let $\preceq$ be a quasi order in the edges of F with no infinite strictly descending sequence and such that $e \preceq f$ whenever f is $\lambda-$linked to e. If the edges of F are not well-quasi-ordered by $\preceq$ then there exists an infinite antichain A of edges of F such that the set $u_F(A)$ is well-quasi ordered by $\preceq$.

Proof. Towards a contradiction we suppose that the lemma does not hold. Let

$$n = \min\{n \in \mathbb{N} | F, \lambda, n \text{ are forming a counterexample}\}$$

Note, that from the choice of our counterexample it follows that:

1. Any $n-$edge labeled forest with no label equal to 0 satisfies the lemma, since otherwise we could form a $(n-1)-$edge labeled counterexample.
2. Moreover, any $n-$edge labeled forest in which the edges labeled 0 are well-quasi-ordered by $\preceq$ it satisfies the lemma, since otherwise by deleting these edges we could get a $n-$edge labeled counterexample for the lemma in which there is no edge labeled 0 and hence a $(n-1)-$edge labeled counterexample.

Let N be the set of edges in F with label 0, from the above observation it follows that N is not well quasi-ordered by $\preceq$ and hence from Lemma 2.4.2 we may consider a minimal bad sequence in N. Let $(a_n)_{n\in\mathbb{N}}$ be such a sequence, that is,

- Badness: $(a_n)_{n\in\mathbb{N}}$ is a bad sequence in N w.r.t $\preceq$, and
- Minimality: $(\forall k \in \mathbb{N})[(e \in N) \wedge (e \preceq a_k) \Rightarrow$ there is no bad sequence in N with $a_1, \ldots, a_{k-1}, e$ as its initial segment]

Since from our hypothesis does not exist an infinity strictly descending sequence w.r.t $\preceq$, it follows that infinite many elements of the sequence $(a_n)_{n\in\mathbb{N}}$ are pairwise incomparable w.r.t $\prec$, and hence, the set of those elements forms an infinite antichain A in $(a_n)_{n\in\mathbb{N}}$. From our assumption that the lemma does not hold it follows that $u_F(A)$ is not well-quasi-ordered by $\preceq$ and hence $u_F(\{a_n | n \in \mathbb{N}\})$ is not well-quasi-ordered by $\preceq$.

We now form another counterexample which would then help us get the desired contradiction. Let R be the maximal -with respect to the number of its edges- subforest of F with all root edges in $u_F(\{a_n | n \in \mathbb{N}\})$.

Claim 3.4.10. *The rooted forest R with the $n-$edge labeling function λ constrained in $E(R)$, consists a counterexample for our lemma.*

Proof of claim. Actually R inherits this property from F.

- Note that $u_F(\{a_n | n \in \mathbb{N}\}) \subseteq E(R)$, and hence the set $E(R)$ is not well-quasi-ordered by the relation $\preceq$, because that would imply that the set $u_F(\{a_n | n \in \mathbb{N}\})$ is well-quasi-ordered by $\preceq$, which is not true.
- In addition, from our assumption that F with the $n-$edge labeling function λ consists a counterexample for the lemma, and the fact that from the construction of the rooted forest R, we have $(\forall B \subseteq E(R))[u_R(B) = u_F(B)]$, it follows that for every infinite antichain $B \subset E(R)$, the set $u_R(B)$ is not well-quasi-ordered by the relation $\preceq$.

□

Since R is a counterexample, the set of edges labeled with 0 in R is not well-quasi-ordered by the relation $\preceq$ and hence by Lemma 2.4.2 we can find an infinite bad sequence in it. Let $(b_n)_{n\in\mathbb{N}} \subseteq E(R) \cap N$ be such a sequence. From the construction of R it follows that:

$$(\forall j \in \mathbb{N})(\exists! s(j) \in \mathbb{N})[b_j \preceq a_{s(j)}]$$

Let $l \in \mathbb{N}$, be such that, $s(l) = \min\{s(j) | (b_j \preceq a_{s(j)}) \wedge (j \in \mathbb{N})\}$. From the minimality of $(a_n)_{n\in\mathbb{N}}$ and the fact that $(b_l \in N) \wedge (b_l \preceq a_{s(l)})$, it follows that the sequence

$$a_1, \ldots, a_{s(l)-1}, b_l, b_{l+1}, \ldots$$

is a good sequence and hence has at least a good pair. Let us consider such a pair: Since $(a_n)_{n\in\mathbb{N}}$ is a bad sequence such a pair cannot have the form (a_i, a_j), but neither the form (a_i, b_j) because for each $j \geq l$ and each $i \leq s(l) (\leq s(j))$ we have that $b_j \preceq a_{s(j)}$ and thus if $a_i \preceq b_j$ by the transitivity of $\preceq$ would imply $a_i \preceq a_{s(j)}$, i.e a good pair in the bad sequence $(a_n)_{n\in\mathbb{N}}$. Thus, the good pair of the above sequence must have the form (b_i, b_j) but that contradicts to the badness of $(b_n)_{n\in\mathbb{N}}$. By assuming that the lemma does hold we have derive a contradiction, hence the lemma follows. □

3.4.2 Lemma on cubic trees

A binary forest is a rooted orientation of a cubic forest with a distinction between left and right outgoing edges.

Definition 3.4.11 (Binary forest)**.** A *binary forest* is a triple (F, l, r) in which F is a rooted forest where the roots have outdegree 1 and l, r are functions defined on nonleaf edges of F, such that the head of each nonleaf edge e of F has exactly two outgoing edges, namely $l(e)$ and $r(e)$.

Lemma 3.4.12 (Lemma on cubic trees)**.** Let (F, l, r) be an infinite binary forest with an $n-$edge labeling λ. Moreover, let $\preceq$ be a quasi-order on the edges of F with no infinite strictly descending sequences, such that $e \preceq f$ whenever f is $\lambda-$linked to e. If the leaf edges of F are well-quasi-ordered by $\preceq$ but the root edges of F are not, then F contains an infinite sequence $(e_n)_{n\in\mathbb{N}}$ of nonleaf edges such that:

(i) $\{e_0, e_1, \ldots\}$ is an antichain with respect to $\preceq$

(ii) $l(e_0) \preceq \ldots \preceq l(e_{i-1}) \preceq l(e_i) \preceq \ldots$

(iii) $r(e_0) \preceq \ldots \preceq r(e_{i-1}) \preceq r(e_i) \preceq \ldots$

Proof. Since the set of root edges of F is not well-quasi-ordered by the relation $\preceq$, neither the set of all edges of F is. Hence by Lemma 3.4.9 it follows the existence of an infinite antichain A of edges, such that the set $u_F(A)$ is well-quasi-ordered by the relation $\preceq$.
Note that since the set of leaf edges of F is well-quasi-ordered, the set A contains finitely many leaf edges of F, because otherwise we could form in A an infinite sequence of leaf edges of F and thus we could find in A a good pair of edges (w.r.t $\preceq$) contradicting to fact that A is an antichain of edges. Hence if A contains any leaf edges of F we can omit them and deduce an infinite antichain $A' \subseteq A$ that does contain any leaf edges. In this case $u_F(A') \subseteq u_F(A)$ and thus $u_F(A')$ is well-quasi-ordered by the relation $\preceq$.
Thus, without loss of generality we may assume that A contains no leaf edges. We now proceed to the construction of the desired sequence of edges.

Since F is a binary forest, by definition the head of each nonleaf edge e of F is the tail of exactly two edges in F, the edges $l(e), r(e)$. Hence $u_F(A) = \{l(e)|e \in A\} \cup \{r(e)|e \in A\}$. Let $(e_n)_{n\in\mathbb{N}}$ be an arbitrary but fixed sequence in A. Provided that $u_F(A)$ is well-quasi-ordered by the relation $\preceq$, and that $(l(e_n))_{n\in\mathbb{N}}$ is an infinite sequence in $u_F(A)$, by Theorem 2.1.7 it follows that $(l(e_n))_{n\in\mathbb{N}}$ has an infinite increasing $(w.r.t \preceq)$ subsequence. Let $(l(e_{k_n}))_{n\in\mathbb{N}}$ be such a sequence. For the same reason, the sequence $(r(e_{k_n}))_{n\in\mathbb{N}}$ has an infinite increasing $(w.r.t \preceq)$ subsequence. Let $(r(e_{k_{l_n}}))_{n\in\mathbb{N}}$ be such a sequence. Then $(l(e_{k_{l_n}}))_{n\in\mathbb{N}}$ is a subsequence of $(l(e_n))_{n\in\mathbb{N}}$ and hence it is also increasing $(w.r.t \preceq)$. It follows that the sequence $(e_{k_{l_n}})_{n\in\mathbb{N}}$ is the desired sequence of nonleaf edges. □

3.4.3 Well-quasi-ordering graphs with bounded branch-width

Definition 3.4.13 (rooted graph)**.** We define a *rooted graph*, to be an ordered pair (G, X), such that G is a graph and X is a subset of the vertex set of G.

Definition 3.4.14 (the minor relation on rooted graphs)**.** Given two rooted graphs, say $(G', X'), (G, X)$, the rooted graph (G', X') will be said to be a *minor* of the rooted graph (G, X) if and only if the following hold:

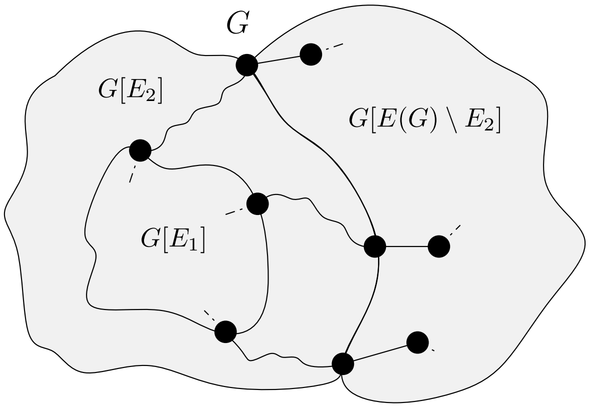

Figure 3.4.1: Illustration of the assumption of Theorem 3.4.16. In the case that is illustrated we have $\gamma_G(E_1) = \gamma_G(E_1, E(G) \setminus E_2) = \gamma_G(E_2) = 3$.

- G' is a minor of G obtained by deletion of edges, deletion of vertices not in X and by contracting edges;
- X' equals with X except if an edge $\{u, v\} \in E(G)$ with $\{u, v\} \cap X \neq \emptyset$ contracted during the deduction of G' as a minor of G, in which case for every such an edge $X' = \{X' \setminus \{u, v\}\} \cup \{v_{new}\}$ where v_{new} is the contraction vertex.

Observation 3.4.15. The minor relation on rooted graphs is clearly a quasi-order with no infinite strictly decreasing sequences.

Theorem 3.4.16. Let G be a graph, and $E_1, E_2 \subseteq E(G)$ such that: $E_1 \subseteq E_2$. We denote by G_1, G_2 the subgraphs of G induced by E_1, E_2 respectively. If $\gamma_G(E_1) = \gamma_G(E_1, E(G) \setminus E_2) = \gamma_G(E_2)$, then the rooted graph $(G_1, \Gamma_G(E_1))$ is a minor of the rooted graph $(G_2, \Gamma_G(E_2))$.

Proof. By Menger's Theorem 1.2.73, the graph induced by $E_2 \setminus E_1$ contains a collection of $\gamma_G(E_1, E(G) \setminus E_2)$ vertex disjoint paths from the set of vertices $\Gamma_G(E_1)$ to the set $\Gamma_G(E_2)$. Contracting these paths in $(G_2, \Gamma_G(E_2))$ and deleting all remaining edges in $E_2 \setminus E_1$ yields $(G_1, \Gamma_G(E_1))$. □

Observation 3.4.17. Let $(G_1, X_1), (G_2, X_2)$ be two rooted graphs such that $|X_1| = |X_2|$, and let also H_1, H_2 be two graphs that are both obtained from $(G_1, X_1), (G_2, X_2)$ by identifying the vertices in X_1 one-to-one with the vertices of X_2. The graphs H_1, H_2 may be non-isomorphic (depending of which vertices identified in the construction of each of them), however, up to isomorphism, there are only finitely many graphs -at most $|X_1|!$- that can be obtained by such an identification. That is the crux of the proof of Theorem 3.4.2.

We are now ready to prove the main result of this section.

Proof of Theorem 3.4.2. Towards a contradiction, we suppose that there exists $n \in \mathbb{N}$ such that, the set of graphs with branch-width at most n - denoted by $\mathcal{G}_n$- is not well-quasi-ordered by the minor relation.

Recall that from Theorem 3.3.37, every graph G has a linked branch-decomposition of width $\mathbf{bw}(G)$, hence for each $G \in \mathcal{G}_n$ we can choose a linked branch-decomposition T_G of G that has width $\mathbf{bw}(G) \leq n$. Without loss of generality, we may assume that each T_G has at least one leaf that does not correspond to any edge of G, since otherwise we can deduce such a branch-decomposition from T_G by subdividing an edge of it and add a pendant edge to make it cubic again.

We now "transform" each of the chosen branch-decompositions to a rooted tree as follows:

Fix an unlabeled leaf r and orient the edges of the branch-decomposition in such a way that all the vertices except r have in-degree 1.

Let (F, l, r) be the rooted binary forest composed of the rooted cubic trees $\{T_G | G \in \mathcal{G}_n\}$. Given an edge e of F, let G be such, that $e \in E(T_G)$, we denote: by E^e the set of edges of G displayed by the component $T_G \setminus e$ that does not contain the root of T_G, by G^e the subgraph of G induced by the set of edges E^e and by X^e the set $\Gamma_G(E^e)$. Moreover, we define a function λ on the set of edges of F such that: $(\forall e \in E(F))[e \in T_G \Rightarrow \lambda(e) = \gamma_G(E^e)]$. Notice that since for each $G \in \mathcal{G}_n$ the width of T_G is $\leq n$, the function λ is an $n-$edge labeling of F. We also define a binary relation $\preceq$ on the set of edges of F, as follows: If e, f are edges of F, then $e \preceq f$ if and only if the rooted graph (G^f, X^f) is a minor of the rooted graph (G^e, X^e).

Claim 3.4.18. *The binary relation $\preceq$ is a quasi-order on the edges of F with no infinite strictly decreasing sequences.*

Proof of Claim 3.4.18. Immediate form Observation 3.4.15. □

Claim 3.4.19. *For any two edges e, f of the rooted binary forest (F, l, r), if f is λ-linked to e, then $e \preceq f$.*

Proof of Claim 3.4.19. Let e, f be two arbitrary but fixed edges of the rooted binary forest (F, l, r) such that f is λ-linked to e. Since f is λ-linked to e, there exist a graph G such that $e, f \in T_G$ and a directed path P on T_G which is starting with e and is ending with f such that $(\forall g \in E(P))[\lambda(g) \geq \lambda(e) = \lambda(f)]$. Notice that $E^f \subseteq E^e$.

Recall Definition 3.3.19 and notice that since T_G is a linked branch-decomposition of G it follows that the edges e, f are linked and thus there exists an edge $g \in E(P)$ such that $\lambda(g) = \gamma_G(E^f, E(G) \setminus E^e)$, thus

$$(\gamma_G(E^e) = \lambda(e) \leq \gamma_G(E^f, E(G) \setminus E^e)) \wedge (\gamma_G(E^f) = \lambda(f) \leq \gamma_G(E^f, E(G) \setminus E^e))$$

and since $(\gamma_G(E^e) \geq \gamma_G(E^f, E(G) \setminus E^e)) \wedge (\gamma_G(E^f) \geq \gamma_G(E^f, E(G) \setminus E^e)))$ and $\gamma_G(E^e) = \lambda(e) = \lambda(f) = \gamma_G(E^f)$, it follows that $\gamma_G(E^e) = \gamma_G(E^f, E(G) \setminus E^e) = \gamma_G(E^e)$.

Hence, by Theorem 3.4.16 it follows that the rooted graph (G^f, X^f) is a minor of the rooted graph (G^e, X^e) and thus $e \preceq f$. Since e, f were two arbitrary edges of (F, l, r) the proof of the claim is complete. □

As can be easily checked the leaf edges of the rooted binary forest (F, l, r) are well-quasi-ordered by the relation $\preceq$, as each of them corresponds to a rotted graph with at most one edge.

Claim 3.4.20. *The root edges of the rooted binary forest (F, l, r) are not well-quasi-ordered by the relation $\preceq$.*

Proof of Claim 3.4.20. The root edges of F are not well-quasi-ordered by $\preceq$ as the associated rooted graphs are the graphs $\{(G, \emptyset) | G \in \mathcal{G}_n\}$. □

By Claims 3.4.18, 3.4.19, 3.4.20 and the fact that the leaf edges of the rooted binary forest (F, l, r) are well-quasi-ordered by the relation $\preceq$ it follows that the rooted binary forest (F, l, r) with the n-edge labeling λ and the quasi-order $\preceq$ meets the requirements of Lemma 3.4.12, and thus there exist an infinite sequence $(e_n)_{n\in\mathbb{N}}$ of non-leaf edges of F such that:

(i) $\{e_0, e_1, \ldots\}$ is an antichain with respect to $\preceq$;

(ii) $l(e_0) \preceq \ldots \preceq l(e_{i-1}) \preceq l(e_i) \preceq \ldots$;

(iii) $r(e_0) \preceq \ldots \preceq r(e_{i-1}) \preceq r(e_i) \preceq \ldots$.

Let $i \in \mathbb{Z}^+$ be an arbitrary but fixed positive integer. Consider the non-leaf edge e_i of the above sequence and let T_G be the cubic tree that e_i belongs. Since T_G has width at most n, it follows that for each set of edges $A \subseteq E(G)$ displayed by an edge in T_G, $\gamma_G(A) \leq n$. Hence $\gamma_G(E^{l(e_i)}) \leq n$ and $\gamma_G(E^{r(e_i)}) \leq n$, thus each of the sets of vertices $X^{l(e_i)}, X^{r(e_i)}$ has at most n elements. As i was arbitrary, it follows that $(\forall i \in \mathbb{N})[(|X^{l(e_i)}| \leq n) \wedge (|X^{r(e_i)}| \leq n)]$, thus by taking a subsequence of $(e_n)_{n\mathbb{N}}$ (if needed), we may assume that the sets $\{X^{l(e_i)} | i \in \mathbb{N}\}$ all have the same cardinality, say $w_L \in \{1, \ldots, n\}$, and also the sets $\{X^{r(e_i)} | i \in \mathbb{N}\}$ all have the same cardinality, say $w_R \in \{1, \ldots, n\}$.

Consider the following two sets, whose elements will be called *labels*:

(i) The set $\{1^{\text{left}}, \ldots, w_L^{\text{left}}\}$, whose elements will be called *left labels*;

(ii) The set $\{1^{\text{right}}, \ldots, w_R^{\text{right}}\}$, whose elements will be called *right labels*.

Recall that $l(e_0) \preceq \ldots \preceq l(e_{i-1}) \preceq l(e_i) \preceq \ldots$ and note that for each $i \in \mathbb{N}$ we can assign a left label to each vertex of the set $X^{l(e_i)}$ in such a way, that: For each $i, j \in \mathbb{N}$ with $i < j$, the graph $G^{l(e_i)}$ can be obtained as a minor of the graph $G^{l(e_j)}$ in such a way that a vertex of the set $X^{l(e_j)}$ "goes" to the vertex of the set $X^{l(e_i)}$ with the same left label. Since $r(e_0) \preceq \ldots \preceq r(e_{i-1}) \preceq r(e_i) \preceq \ldots$, we can assign right labels to the vertices of each set of $\{X^{r(e_i)} | i \in \mathbb{N}\}$ in a similar way.

What we are trying to prove is that there exist positive integers i, j with $i < j$ such that the rooted graph (G^{e_i}, X^{e_i}) is a minor of the rooted graph (G^{e_j}, X^{e_j}) and derive by this way the desired contradiction. Because in that case we have $e_i \preceq e_j$ contradicting to the fact that the set $\{e_0, e_1, \ldots\}$ is an antichain with respect to $\preceq$. We need to show that, for some $i < j$, the two pairs "glue" together in the same way. The understanding of Observation 3.4.17 is the key for the understanding of the rest part of the proof.

Vertices in $X^{l(e_i)} \cap X^{r(e_i)}$ get both left and right labels. Since there are only finitely many different subsets of $\{1^{\text{left}}, \ldots, w_L^{\text{left}}\}$, infinitely often it is the same one that gives the lefts labels in $X^{l(e_i)} \cap X^{r(e_i)}$. Of these , infinitely often its is the set of right labels. Thus, we may relabel (going to this subsequence) in such a way that, for each i, the set of left labels and the set of right labels occurring in $X^{l(e_i)} \cap X^{r(e_i)}$ are always the same.

Notice that $X^{e_i} \subseteq X^{l(e_i)} \cap X^{r(e_i)}$. Thus, every vertex of X^{e_i} has a label: some will have only left-labels, some will have only right-labels, and other will have both. Again, there are only finitely many possibilities for which labels can appear and for which combinations. Thus, we can assume that

(i) The sizes of the sets X^{e_i} are all the same;

(ii) the set of left-labels in the sets X^{e_i} is always the same;

(iii) the set of right-labels in the sets X^{e_i} is always the same.

Note that (ii) and (iii) combine to show that the set of left-only labels is always the same, and that the set of right-only labels is always the same, because if a vertex has both left-label and right-label, then these two labels always go together.

Now, for $i < j$,

$$\big(G^{l(e_i)}, X^{l(e_i)}\big) \text{ is a minor of } \big(G^{l(e_j)}, X^{l(e_j)}\big)$$

and

$$\big(G^{r(e_i)}, X^{r(e_i)}\big) \text{ is a minor of } \big(G^{r(e_j)}, X^{r(e_j)}\big)$$

We now show that $\big(G^{e_i}, X^{e_i}\big)$ is a minor of $\big(G^{e_j}, X^{e_j}\big)$.
The graph G^{e_i} is obtained from $G^{l(e_i)}$ and $G^{r(e_i)}$ by identifying the vertices in $X^{l(e_i)} \cap X^{r(e_i)}$; this identification is the same in both G^{e_i} and G^{e_j}. □

3.5 Planar graphs

The purpose of this section is to present the proof of the following special case of Robertson and Seymour's theorem.

Theorem 3.5.1 (Robertson and Seymour [106]). If $G_1, G_2, \dots$ is any infinite sequence of planar graphs, then there exist i, j with $j > i \geq 1$ such that G_i is isomorphic to a minor of G_j.

The main ingredient that we will need is the following:

Theorem 3.5.2 (Robertson and Seymour [106]). For any planar graph Γ the set of all planar graphs we no minor isomorphic to Γ is well-quasi-ordered by the minor relation.

The above have the following immediate corollary.

Corollary 3.5.3. Let $\mathcal{P}$ be any set of planar graphs, then $\mathcal{P}$ is well-quasi-ordered by the minor relation.

Definition 3.5.4 (Explicit definition of the $(k \times k)$-*grid* graph). Let $k \geq 1$ be a positive integer, then the $(k \times k)$-*grid* is the graph:

$$\big(\{1, \dots, k\}^2, \{\{(x_1, y_1), (x_2, y_2)\} : |x_1 - x_2| + |y_1 - y_2| = 1\}\big).$$

Notation 3.5.5. We denote the $(k \times k)$-grid by Λ_k.

Definition 3.5.6 (Recursive definition of the grid graph). Let $k \geq 1$ be a positive integer, then the $(k \times k)$-*grid* Λ_k, is defined recursively as follows:

- For $k \in \{1, 2\}$ $\Lambda_k = \big(\{1, \dots, k\}^2, \{\{(x_1, y_1), (x_2, y_2)\} : |x_1 - x_2| + |y_1 - y_2| = 1\}\big)$.
- For $k \geq 3$, Λ_k is the graph that we obtain from Λ_{k-2}, if we:
  (i) rename each of the vertices and the corresponding edges of Λ_{k-2}, as follows: for each $i, j \in \{1, \dots, k-2\}$ the vertex (i, j) is renamed to $(i + 1, j + 2)$;

(ii) add the $4k-4$ vertices $\{(1,i),(i,1),(i,k),(k,i)|1 \leq i \leq k\}$ to the vertex set of Λ_{k-2};

(iii) add the edges $\{\{(1,i),(1,i+1)\}|1 \leq i \leq k-1\} \cup \{\{(i,k),(i+1,k)\}|1 \leq i \leq k-1\} \cup \{\{(k,i),(k,i+1)\}|1 \leq i \leq k-1\} \cup \{\{(i,1),(i+1,1)\}|1 \leq i \leq k-1\}$

(iv) add the edges $\{\{(1,i),(2,i)\}|2 \leq i \leq k-1\} \cup \{\{(i,k),(i,k-1)\}|2 \leq i \leq k-1\} \cup \{\{(k,i),(k-1,i)\}|2 \leq i \leq k-1\} \cup \{\{(i,1),(i,2)\}|2 \leq i \leq k-1\}$.

Notation 3.5.7. Let Π_k be a statement concerning Λ_k, we denote by $\Pi_k(T)$ the fact that Π_k is true.

Theorem 3.5.8 (Structural induction scheme for the grid graph)**.** Let k_0 be a positive integer and let Π_k be a statement concerning Λ_k, such that:

(i) $\Pi_{k_0}(T)$ and $\Pi_{k_0+1}(T)$;

(ii) $(\forall k \geq k_0+2)[\Pi_{k-2}(T) \Rightarrow \Pi_k(T)]$.

Then $(\forall k \geq k_0)[\Pi_k(T)]$

Proof. Immediate by the usual induction on integers. □

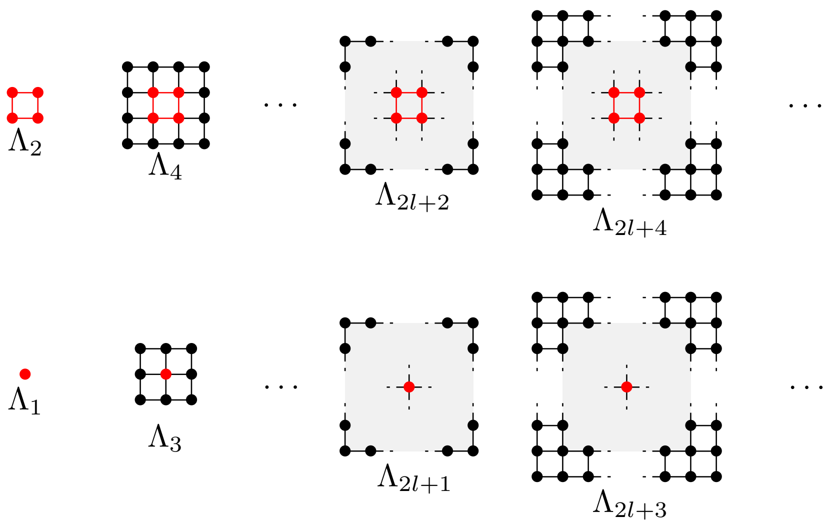

Figure 3.5.1: Illustration of the recursive definition of the grid graph (Definition 3.5.6).

Robertson, Seymour, and Thomas [115] proved that every planar graph is isomorphic to a minor of a large enough grid.

Theorem 3.5.9 (Robertson, Seymour, and Thomas [115])**.** If Γ is a planar graph with $|V(\Gamma)| + 2|E(\Gamma)| \leq n$, then H is isomorphic to a minor of the $(2n \times 2n)$-grid.

Let Γ be an arbitrary but fixed planar graph the transitivity of the minor relation on graphs and the above theorem implies that when a planar graph, say Π has no minor isomorphic to Γ there is a positive integer k, such that Π has not a minor isomorphic to the $(k \times k)$-grid. Hence we may deduce informations about the "rough" structure of planar graphs which exclude a fixed planar graph as a minor, by studying the case in which the excluded minor is a grid.

This case is studied on the Excluded Grid Theorem for planar graphs (Theorem 3.5.24) which is one of the two basic ingredients for the proof of Theorem 3.5.2. Informally, the Excluded Grid Theorem for planar graphs states that if a planar graph has large branch-width then it contains a large grid as a minor. Thus if a planar graph excludes a grid as a minor then it has bounded branch-width.

Hence, for any planar graph Γ, the set of all planar graphs with no minor isomorphic to Γ is a set of graphs with bounded branch-width. Here comes the second ingredient which is Theorem 3.4.3 which states that any set of graphs with bounded branch-width is well-quasi-ordered by the minor relation and has already been proved in Subsection 3.4.3.

3.5.1 Nooses and Θ-triples

Definition 3.5.10 (Jordan curve - Simple closed curve in the plane)**.** A *Jordan curve* is the image of a continuous function $\phi : [0, 1] \to \mathbb{R}^2$ such that $\phi(0) = \phi(1)$ and the restriction of ϕ to $[0, 1)$ is injective.

Theorem 3.5.11 (Jordan curve theorem)**.** The complement of a simple closed curve in the plane has exactly two connected components. The one of those is bounded and the other one unbounded.

Definition 3.5.12 (Noose of a plane graph)**.** Let Γ be a plane graph and let N be a Jordan curve. We say that a Jordan curve N is a noose of Γ when $N \cap \Gamma \subseteq V(\Gamma)$, i.e., N does not intersect any of the edges of Γ. We denote by $V(N)$ the set $N \cap \Gamma$ and by $|N|$ the order of $V(N)$.

Definition 3.5.13 (*I-arc* of a plane graph)**.** Let Γ be a plane graph an *I-arc* of Γ is a subset I of $\mathbb{R}^2$ that is homeomorphic to the open interval $(0, 1)$ in $\mathbb{R}^2$, does not intersect the edges of Γ and, moreover, there exist two vertices x and y of G, called *endpoints* of I, where $I \cup \{x\} \cup \{y\}$ is homeomorphic to the closed interval $[0, 1]$ of $\mathbb{R}^2$ (notice that an arc may intersect vertices of Γ). We also say that I is an arc *between* x and y and we use the notation $V(I) = I \cup \Gamma$ and $|I| = |V(I)| + 2$.

Observation 3.5.14. Let Γ be a plane graph and let N be a noose of Γ. Then the connected components of the set $N \setminus \Gamma$ are $|N|$ I-arcs where the endpoints of each such arc are vertices of G that appear consecutively on N.

Definition 3.5.15 (*Open disks bounded by N)***.** Let Γ be a plane graph and let N be a noose of Γ. From the Jordan curve Theorem 3.5.11 it follows that $\mathbb{R}^2 \setminus N$ has two connected components, that are open disks. We call them *open disks bounded by* N.

Definition 3.5.16 (A noose separates two vertices or two edge)**.** Let Γ be a plane graph and let N be a noose of Γ. If x and y are vertices of $\Gamma \setminus V(N)$ or edges of Γ, we say that N *separates* x and y if they are in different open disks bounded by N.

Definition 3.5.17 (Equivalent nooses)**.** Two nooses N_1 and N_2 of Γ are *equivalent* if $N_1 \cap N_2 = V(N_1) = V(N_2)$ both N_1 and N_2 meet their vertices in the same cyclic ordering and for each pair I, I' of connected components of $(N_1 \cup N_2) \cap (N_1 \cap N_2)$ with the same endpoints, it holds that both I and I' are subsets of the same face of Γ.

Definition 3.5.18 (Θ-triple of a plane graph)**.** Let Γ be a plane graph and let N_1, N_2, and N_3 nooses of Γ. We say that (N_1, N_2, N_3) form a Θ-triple of Γ when $N_1 \cap N_2 \cap N_3$ is a set consisting of two

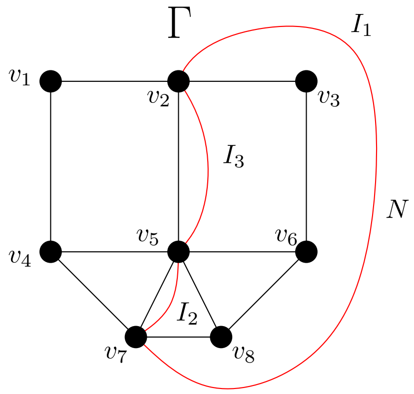

Figure 3.5.2: Illustration of Definitions 3.5.12, 3.5.13. The simple closed curve N drawn by red color is a noose of the plane graph Γ such that $N \cap \Gamma = V(N) = \{v_2, v_5, v_7\} \subseteq V(\Gamma)$. The three I-arcs I_1, I_2, I_3 are the connected components of $N \setminus \Gamma$.

vertices of Γ and $\mathbb{R}^2 \setminus (N_1 \cup N_2 \cup N_3)$ has three connected components. If each of these components contains an edge of Γ, we say that (N_1, N_2, N_3) is *proper*. That way we say that any proper Θ-triple (N_1, N_2, N_3) generates a 3-partition $\{E_1, E_2, E_3\}$ of $E(\Gamma)$ such that E_i is the set of the edges contained in the open disk bounded by N_i that does not contain the arc $(N_1 \cup N_2 \cup N_3) \setminus N_i$.

Observation 3.5.19. Let N_1, N_2, and N_3 be a Θ-triple of a graph Γ and let $I = N_1 \cap N_2$. Then $|N_1| + |N_2| = |N_3| + 2 \cdot |I| - 2$.

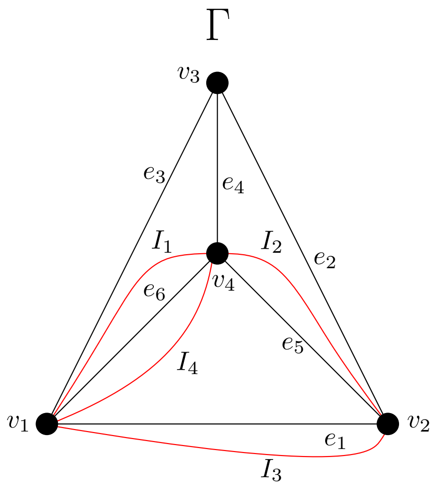

Figure 3.5.3: A proper Θ-triple of a plane graph.
Let $N_1 = I_2 \cup I_3 \cup I_4 \cup \{v_1, v_2, v_4\}, N_2 = I_1 \cup I_4 \cup \{v_1, v_4\}$ and
$N_3 = I_1 \cup I_2 \cup I_3 \cup \{v_1, v_2, v_4\}$, then (N_1, N_2, N_3) is a proper Θ-triple of Γ that induce the 3-partition $\{E_1, E_2, E_3\}$ of $E(\Gamma)$, where $E_1 = \{e_6\}, E_2 = \{e_1, e_5\}, E_3 = \{e_2, e_3, e_4\}$.

Lemma 3.5.20. Let Γ be a connected plane graph, and let x and y be two non-adjacent vertices of Γ. Let also S be a minimum (x, y)-separator of Γ. Then Γ has a noose N where $V(N) = S$.

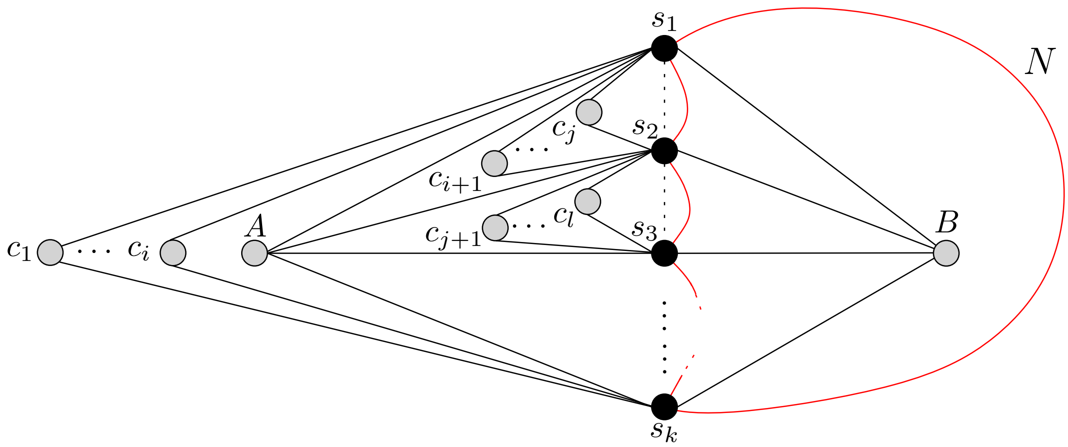

Figure 3.5.4: The positions of the connected components of $\Gamma \setminus S$ and the noose N in *Case 3* of the proof of Lemma 3.5.20.

Proof. We distinguish the following cases:

Case 1: $|S| = 1$. Given any vertex $v \in V(\Gamma)$, it is trivial to consider a noose N of Γ such that $V(N) = \{v\}$. Just consider one of the faces of Γ that have in their boundary the vertex v and draw in there the noose N.

Case 2: $|S| = 2$. We call s_1, s_2 the two vertices of S. Consider on Γ the connected components of the graph $\Gamma \setminus S$, contract each edges whose both endpoints are inside in one of those connected components. On the resulting graph we call A/B the vertex that corresponds to the connected component of $\Gamma \setminus S$ that contains the vertex x/y respectively, we also call $c_1, \dots c_m$ the vertices that corresponds to the other connected components of $\Gamma \setminus S$.

Observe, that for each $i \in \{1, \dots, m\}$ in the resulting graph at most two edges could be incident to the vertex c_i, one that has as its one endpoint the vertex s_1 and one that has as its one endpoint the vertex s_2. This structure allows us to consider a new planar drawing of Γ which would then make easy the drawing of the desired noose N.

Let D be the bounded disc that is defined by the cycle $\{A, s_1\} \cup \{s_1, B\} \cup \{B, s_2\} \cup \{s_2, A\}$. We may transfer each of the connected components $c_1, \dots, c_m$ inside D and then draw the desired noose N, in the way that is illustrated in Figure 3.5.5.

Case 3: $|S| \geq 3$. Consider on Γ the connected components of the graph $\Gamma \setminus S$, contract each edges whose both endpoints are inside in one of those connected components. We call Γ' the resulting graph. Clearly Γ' is a minor of Γ. On Γ' we call A/B the vertex that corresponds to the connected component of $\Gamma \setminus S$ that contains the vertex x/y respectively, we also call $c_1, \dots c_m$ the vertices that corresponds to the other connected components of $\Gamma \setminus S$. Let $s_1, s_2, \dots s_k$ be an enumeration of the vertices of S, such that if $i_1 < j_1$ and $i_2 < j_2$ then the cycle $\{A, s_{j_1}\} \cup \{s_{j_1}, B\} \cup \{B, s_{i_2}\} \cup \{s_{i_2}, A\}$ is inside the close disc that is defined by the cycle $\{A, s_{i_1}\} \cup \{s_{i_1}, B\} \cup \{B, s_{j_2}\} \cup \{s_{j_2}, A\}$.

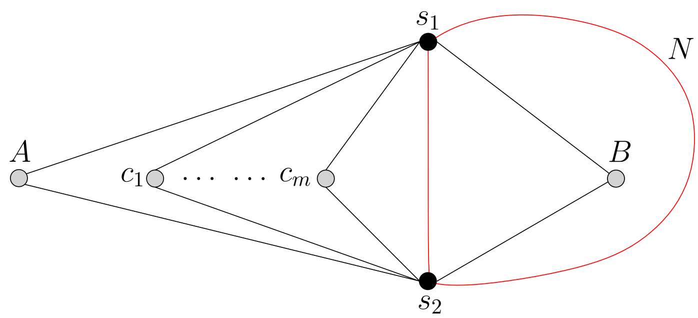

Figure 3.5.5: The positions of the connected components of $\Gamma \setminus S$ and the noose N in *Case 2* of the proof of Lemma 3.5.20.

Claim 3.5.21. *Let $i \in \{1, \ldots, m\}$, then there exist at most two edges on Γ' that are incident to the vertex c_i and that have their other endpoint to a vertex in S.*

Proof of Claim 3.5.21. Let us suppose towards a contradiction that there exist $i \in \{1, \ldots, m\}$ such that the vertex c_i is adjacent with at least three vertices, say s_x, s_y, s_z, of the set S. Notice that Menger's Theorem 1.2.73 guaranties the existence of three internally vertex disjoint paths, say P_1, P_2, P_3 in Γ, such that each of those has as its endpoints the vertices x, y and as an internal vertex, the vertex s_x, s_y, s_z respectively. It's now immediate to deduce that Γ has a minor isomorphic to the complete bipartite graph $K_{3,3}$ contradicting to Wagner-Kuratowski's Theorem 3.1.2. □

The following observation relies on the fact that Γ' is plane.

Observation 3.5.22. Let $i \in \{1, \ldots, m\}$, then if c_i is adjacent with two vertices of S, then those vertices are either successionals on the enumeration of S that we have consider either the vertices s_1, s_k.

Thus, by Observation 3.5.22 and Claim 3.5.21 we can consider the drawing of Γ that corresponds to the drawing of Γ' that is illustrated in Figure 3.5.4 and the existence of the noose N follows. □

3.5.2 Sphere-cut decompositions

Definition 3.5.23 ((partial) Sphere-cut decomposition of a plane graph**).** Given a plane graph Γ and a partial branch-decomposition $B = (T, \tau)$ of Γ. We say that B is a *partial sphere-cut decomposition* of Γ if there is a function ω, mapping each edge $e \in E(T)$ to a noose N_e of Γ such that $V(N_e) = \mathbf{mid}(e)$. If B is a complete partial branch-decomposition of Γ then we say that B is *sphere-cut decomposition*

3.5.3 The excluded grid theorem for planar graphs

Our proof of the "Excluding Grid Theorem for planar graphs" is due to an unpublished manuscript of Thilikos [117]. The least bound of the branch-width of a graph that is sufficient to guarantee the containment of a $(k \times k)$-grid minor that we prove in the following theorem, was proved in terms

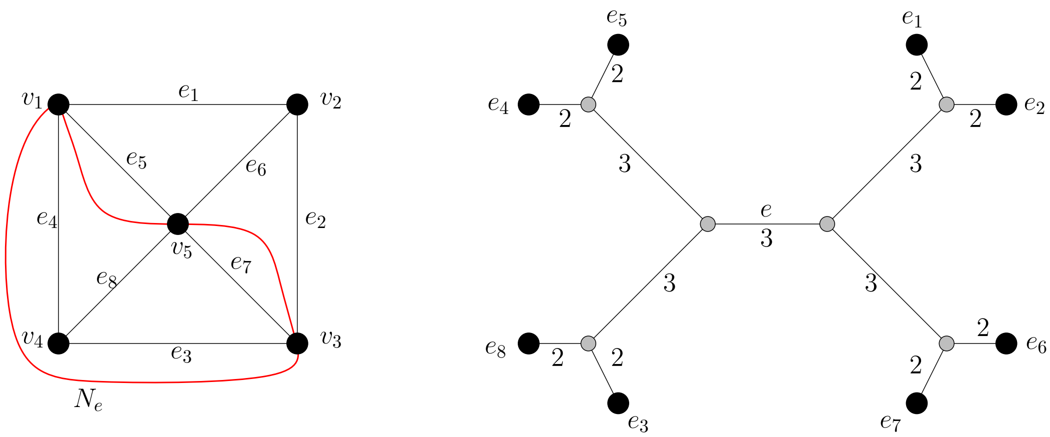

Figure 3.5.6: Illustration of Definition 3.5.23. A planar graph on the left hand side and a sphere-cut decomposition of width 3 of it on the right hand side. The noose N_e -drawn by red color-corresponds to the edge e of the sphere-cut decomposition.

of *tangles* [9] by Robertson, Seymour and Thomas in [115, Theorem 6.3]. A better bound obtained in 2012 by Gu and Tamaki [63].

Theorem 3.5.24 (Excluding Grid Theorem for planar graphs)**.** Let $k \geq 2$ be an integer and Γ be a planar graph on n vertices. If $\mathbf{bw}(\Gamma) > 4k-4$, then Γ contains a $(k \times k)$-minor.

Lemma 3.5.25. Let $k \geq 2$ be an integer. Let Γ be a plane n-vertex graph embedded inside a closed disk D of $\mathbb{R}^2$ with boundary N and such that $S := \Gamma \cap N$ of $4k-4$ vertices of Γ. The clock-wise ordering of the vertices of S on N is $v_1^{\text{up}}, \dots, v_k^{\text{up}} = v_1^{\text{right}}, \dots, v_k^{\text{right}} = v_k^{\text{down}}, \dots, v_1^{\text{down}} = v_k^{\text{left}}, \dots, v_1^{\text{left}} = v_1^{\text{up}}$. Suppose also that Γ is the union of two collections of paths $\mathcal{P}^{\leftrightarrow} = \{P_1^{\leftrightarrow}, \dots, P_k^{\leftrightarrow}\}$ and $\mathcal{P}^{\updownarrow} = \{P_1^{\updownarrow}, \dots, P_k^{\updownarrow}\}$ such that:

(i) for each $i \in \{1, \dots, k\}$, the endpoints of $P_i^{\leftrightarrow}$ are v_i^{left} and v_i^{right} and the endpoints of $P_i^{\updownarrow}$ are v_i^{up} and v_i^{down} ;

(ii) the paths in $\mathcal{P}^{\leftrightarrow}$ are pairwise vertex disjoint and also the paths in $\mathcal{P}^{\updownarrow}$ are pairwise vertex disjoint.

Then Γ contains a Λ_k-minor.

Moreover, there exists an algorithm that given $\mathcal{P}^{\leftrightarrow}$ and $\mathcal{P}^{\updownarrow}$, outputs a minor model of $(k \times k)$-grid in $O(n)$ steps.

Proof. Let Γ be a graph, we denote by $\Gamma_k(T)$ the fact that Γ meets the requirements of Lemma 3.5.25 for the integer k.

In what follows, we prove the following slightly stronger statement:

[9] A tangle is a notion of a highly connected substructure of a graph, which was introduced in [111] and which is not defined in the present work. We refer the interested reader in [33].

Claim 3.5.26. *For each integer $k \geq 2$, the following statement concerning Λ_k is true:*
If Γ is a graph such that $\Gamma_k(T)$ and $\eta : V(\Lambda_k) \to V(\Gamma)$ is a partial function, which is defined as follows:

$\eta\big((1,1)\big) = v_1^{\text{up}}, \ldots, \eta\big((1,k)\big) = v_k^{\text{up}}$,
$\eta\big((1,k)\big) = v_1^{\text{right}}, \ldots, \eta\big((k,k)\big) = v_k^{\text{right}}$,
$\eta\big((k,k)\big) = v_k^{\text{down}}, \ldots, \eta\big((k,1)\big) = v_1^{\text{down}}$*, and*
$\eta\big((k,1)\big) = v_k^{\text{left}}, \ldots, \eta\big((1,1)\big) = v_1^{\text{left}}$.

Then there exists a function $\alpha : V(\Lambda_k) \to V(\Gamma)$, such that $\eta \subseteq \alpha$ and Λ_k is an α-rooted minor of Γ.

Let us denote by Π_k the statement of Claim 3.5.26 for the grid Λ_k. Recall Observation 1.2.54 and notice that we shall have established $\Pi_k(T)$ if we prove the following: There exist a procedure which takes as an input a graph Γ such that $\Gamma_k(T)$ and the correspondents collections of paths $\mathcal{P}^{\leftrightarrow}$, $\mathcal{P}^{\updownarrow}$, and outputs a graph $\tilde{\Gamma}$ for which there exist a function $\alpha : V(\Lambda_k) \to V(\tilde{\Gamma})$ such that:

(i) $\eta \subseteq \alpha$;

(ii) α is an isomorphism between Λ_k and the graph $\tilde{\Gamma}$;

(iii) the graph $\tilde{\Gamma}$ is the result of a sequence of $\alpha(V(\Lambda_k))$-maintaining contractions and edge removals on Γ.

Fix $k \geq 2$ and let Γ be an arbitrary but fixed graph such that $\Gamma_k(T)$. As a first step we set up some terminology for the course of our proof.

We set $C = \{v_1^{\text{up}}, v_1^{\text{right}}, v_k^{\text{down}}, v_k^{\text{left}}\}$ and $L = S \setminus C$. We call the vertices in C *corner* vertices of Γ, the vertices in L *lateral* vertices of Γ and the vertices in Γ that are neither lateral nor corner vertices are called *central* vertices. We also call the edges of the paths in $\mathcal{P}^{\leftrightarrow}$ *horizontal edges* and the edges of the paths in $\mathcal{P}^{\updownarrow}$ *vertical edges*.

Consider the connected components of $D \setminus \Gamma$ that contain points of N. We call these sets *border regions* of Γ (notice that border regions are not open sets). We distinguish four types of border regions:

(i) those that have in their boundary two vertices in $\{v_1^{\text{up}}, \ldots, v_k^{\text{up}}\}$, which we will call *up-regions*,

(ii) those that have in their boundary two vertices in $\{v_1^{\text{right}}, \ldots, v_k^{\text{right}}\}$, which we will call *right-regions*,

(iii) those that have in their boundary two vertices in $\{v_1^{\text{down}}, \ldots, v_k^{\text{down}}\}$,which we will call *down-regions*, and

(iv) those that have in their boundary two vertices in $\{v_1^{\text{left}}, \ldots, v_k^{\text{left}}\}$, which we will call *left-regions*.

We now proceed to the presentation of the procedure that we described above. The procedure consists in the successive application of a series of normalization operations. We remark that in what follows whenever we contract an edge of Γ that its one endpoint, say v, is in S we name the contraction vertex by v.

Normalization 1: Apply the following operation on Γ as long as this is possible: If there is a central vertex of degree 2 such that, only one of the $2k$ paths has edges that have this vertex as an endpoint, then pick one of the two edges that has this vertex as endpoint and contract this edge.

Normalization 2: Apply the following operation on Γ as long as this is possible: If for two paths $P_i^{\updownarrow} \in \mathcal{P}^{\updownarrow}$ and $P_j^{\leftrightarrow} \in \mathcal{P}^{\leftrightarrow}$ there is a connected component Y of $P_i^{\updownarrow} \cap P_j^{\leftrightarrow}$ that is not a single vertex of Γ, then contract all the edges of Y to a single vertex.

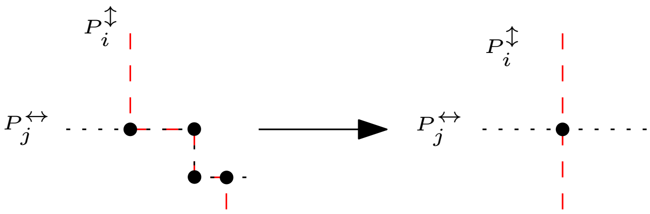

Figure 3.5.7: Illustration of Normalization 2.

Normalization 3: Apply the following operation on Γ as long as this is possible: if some vertex of $\Gamma \cap N$ has degree 1, then contract the edge that has it as an endpoint.

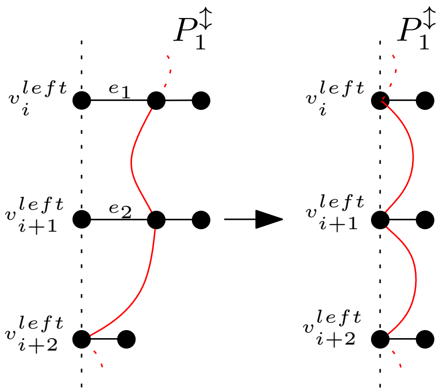

Figure 3.5.8: On the left hand side are illustrated two vertices of $\Gamma \cap N$ $(v_i^{left}, v_{i+1}^{left})$ that have degree 1 and on the right hand side is illustrated the result of the application of the operation that is described on Normalization 3 two times on that part of Γ.

Notice that, as a result of *Normalization 1*, *Normalization 2* and *Normalization 3*, all central vertices of Γ have degree 4, all lateral vertices have degree 3 and all corner vertices have degree 2.

Suppose now that Γ is the result of the above two normalizations and let f be a border region of Γ. We say that f is *regular* if it is an up-region/right-region/down-region/left-region and all the edges of Γ that are in its boundary are edges of $P_1^{\leftrightarrow}/P_k^{\updownarrow}/P_k^{\leftrightarrow}/P_1^{\updownarrow}$ respectively. Otherwise we say that the boundary region f is *irregular*.

Notice that if an up-region/down-region is irregular, this means that its boundary contains at least one vertical edge. Analogously, if a right-region/left-region is irregular, this means that its boundary contains at least one horizontal edge.

The next operation makes all border regions regular and is the following:

Normalization 4: Apply the following operation on Γ as long as this is possible: Let f be an irregular up-region of Γ and let $e = \{x, y\}$ be a vertical edge in the boundary of f. Clearly e is the edge of some path $P_i^{\updownarrow} \in \mathcal{P}^{\updownarrow}$. Notice that x and y are the endpoints of a subpath P of $P_1^{\leftrightarrow}$. Remove from Γ the edges of P. Also update $P_1^{\leftrightarrow}$ by removing from it the edges and the internal vertices of P and adding the edge e in the resulting graph. In case f is an irregular down-region, the operation is defined in the same way by replacing $P_1^{\leftrightarrow}$ by $P_k^{\leftrightarrow}$. Also, in the cases where f is a right-region/left-region, we again copy the above description by replacing $P_1^{\leftrightarrow}$ by $P_k^{\updownarrow}/P_1^{\updownarrow}$ and $P_i^{\updownarrow}$ by $P_i^{\leftrightarrow}$. After each such substitution operation apply *Normalization 1*, *Normalization 2* and *Normalization 3* and proceed until all border regions are regular.

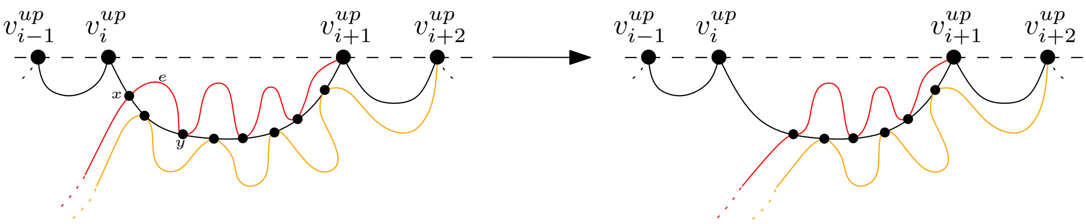

Figure 3.5.9: On the left hand side is illustrated an irregular up-region and on the right hand side it is illustrated the same up-region after one application of the operation that is described on normalization 4.

Let Γ be the graph obtained after *Normalization 4*. We call the faces of Γ that contain at least one corner vertex, *corner faces* of Γ. A corner face is denoted by f_{lu}, f_{ur}, f_{rd}, and f_{dl} if its boundary contains the corner vertex v_1^{left}, v_k^{up}, v_k^{right}, and v_1^{down} respectively. Notice that all edges in f_{lu} are edges of $P_1^{\leftrightarrow}$, $P_2^{\leftrightarrow}$, $P_1^{\updownarrow}$ and $P_2^{\updownarrow}$. We call P_{lu} the maximum subpath of the boundary of f_{lu} whose edges are edges from $P_2^{\leftrightarrow}$ and $P_2^{\updownarrow}$. Notice that the edges in P_{lu} should be alternating edges form $P_2^{\leftrightarrow}$ and $P_2^{\updownarrow}$.

If there are only two edges in P_{lu}, one from $P_2^{\leftrightarrow}$ and one from $P_2^{\updownarrow}$, then we say that f_{lu} is regular. Otherwise f_{lu} is irregular. The regularity of the faces $f_{ur}/f_{rd}/f_{dl}$ is defined by copying the same definition with the difference that now $P_2^{\leftrightarrow}$ and $P_2^{\updownarrow}$ are replaced by: $P_2^{\leftrightarrow}$ and $P_{k-1}^{\updownarrow}$/ $P_{k-1}^{\leftrightarrow}$ and $P_{k-1}^{\updownarrow}$/ $P_{k-1}^{\leftrightarrow}$ and $P_2^{\updownarrow}$ respectively.

The next and final operation of our procedure makes all corner faces regular and is the following:

Normalization 5: If f_{lu} is non-regular, then let $e = \{x, y\} \in E(P_{lu})$. Clearly, e is either an edge of $P_2^{\leftrightarrow}$ or an edge of $P_2^{\updownarrow}$. If $e \in E(P_2^{\leftrightarrow})$, then let P be the subpath of $E(P_2^{\leftrightarrow})$ that has the same endpoints as e. We remove the edges of P from Γ. Then we update $P_2^{\leftrightarrow}$ by removing from it the edges and the internal vertices of P and adding to the resulting graph the edge e. Then we apply *Normalization 1*, *Normalization 2* and *Normalization 3*.

If $e \in E(P_1^{\leftrightarrow})$, we copy the same operation by exchanging the role of $P_2^{\leftrightarrow}$ and $P_2^{\leftrightarrow}$. In the

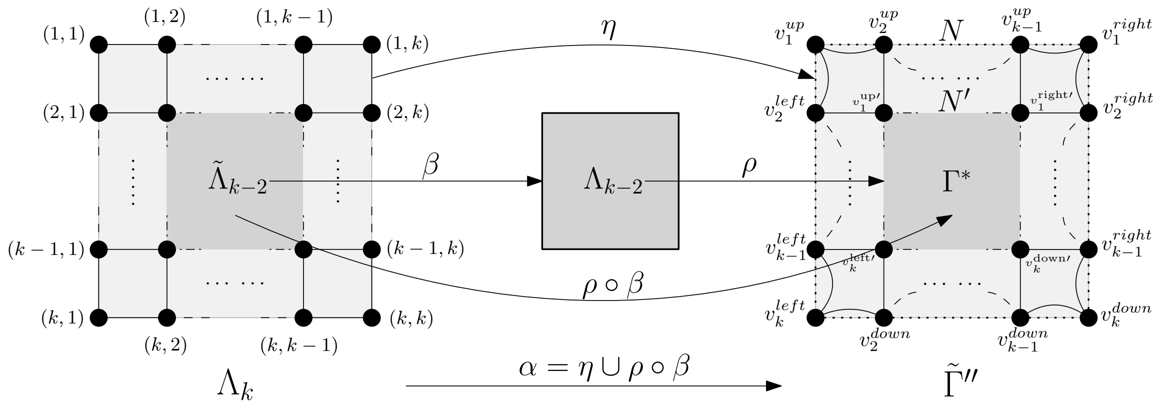

Figure 3.5.10: Illustration of the induction step on the proof of Claim 3.5.26

case where $f_{ur}/f_{rd}/f_{dl}$ is non-regular we copy the above with the difference that now $P_2^{\leftrightarrow}$ and $P_2^{\updownarrow}$ are replaced by $P_{k-1}^{\leftrightarrow}$ and $P_2^{\updownarrow}/P_{k-1}^{\leftrightarrow}$ and $P_{k-1}^{\updownarrow}/P_2^{\leftrightarrow}$ and $P_{k-1}^{\updownarrow}$.

We denote by $\tilde{\Gamma}$ the graph obtained after the above normalizations. Clearly $\tilde{\Gamma}$ is the result of a sequence of $\eta(V(\Lambda_k))$-maintaining contractions and edge removals on Γ.

Proof of Claim 3.5.26. We will prove the claim using the induction scheme described in Theorem 3.5.8.

Induction Basis: For $k = 2$, if Γ is a graph such that $\Gamma_2(T)$ then it is straightforward that:

- $\alpha := \eta$ is an isomorphism between Λ_2 and $\tilde{\Gamma}$;
- the graph $\tilde{\Gamma}$ is the result of $\alpha(V(\Lambda_2))$-maintaining contractions and edge removals on Γ.

Thus the graph $\tilde{\Gamma}$ and the function α witness $\Pi_2(T)$.

For $k = 3$, if Γ is a graph such that $\Gamma_3(T)$ then it is straightforward that the graph $\tilde{\Gamma}$ obtained after the above normalizations has only one central vertex, say v^{center}. Then:

- $\alpha := \eta \cup \{((2,2), v^{\text{center}})\}$ is an isomorphism between Λ_3 and $\tilde{\Gamma}$;
- the graph $\tilde{\Gamma}$ is the result of $\alpha(V(\Lambda_3))$-maintaining contractions and edge removals on Γ.

Thus the graph $\tilde{\Gamma}$ and the function α witness $\Pi_3(T)$.

Induction Hypothesis: Let k be arbitrary but fixed integer such that $k \geq 4$, we suppose $\Pi_{k-2}(T)$.

Induction Step: We now prove $\Pi_k(T)$. Let Γ be a graph such that $\Gamma_k(T)$. Let $\tilde{\Gamma}$ be the graph obtained after executing the aforementioned procedure on Γ. Notice that, because of the above normalizations, all the border regions and corner faces of $\tilde{\Gamma}$ are regular. Notice also that each path in $\mathcal{P}^{\leftrightarrow}$ or $\mathcal{P}^{\updownarrow}$ remained a path with the same endpoints in $\tilde{\Gamma}$ as in Γ. We denote by $\tilde{\mathcal{P}}^{\leftrightarrow} = \{\tilde{P}_1^{\leftrightarrow}, \ldots, \tilde{P}_k^{\leftrightarrow}\}$ and $\tilde{\mathcal{P}}^{\updownarrow} = \{\tilde{P}_1^{\updownarrow}, \ldots, \tilde{P}_k^{\updownarrow}\}$ the resulting collections of paths, whose union is $\tilde{\Gamma}$. Notice also that the graph $\tilde{P}_1^{\leftrightarrow} \cup \tilde{P}_k^{\updownarrow} \cup \tilde{P}_k^{\leftrightarrow} \cup \tilde{P}_1^{\updownarrow}$ is a cycle of $\tilde{\Gamma}$ that contains the whole $\tilde{\Gamma}$ in one of the closed disks that it defines. For $i \in \{1, \ldots, k-2\}$, we denote by $v_i^{\text{up}\prime}/v_i^{\text{right}\prime}/v_i^{\text{down}\prime}/v_i^{\text{left}\prime}$ as the unique neighbor of $v_{i+1}^{\text{up}}/v_{i+1}^{\text{right}}/v_{i+1}^{\text{down}}/v_{i+1}^{\text{left}}$ that is not in the cycle $\tilde{\Gamma}\cap N$. Clearly, by this definition $v_{k-2}^{\text{up}\prime}=v_1^{\text{right}\prime}$, $v_{k-2}^{\text{right}\prime}=v_{k-2}^{\text{down}\prime}$, $v_1^{\text{down}\prime}=v_{k-2}^{\text{left}\prime}$, and $v_1^{\text{left}\prime}=v_1^{\text{up}\prime}$.

Let N' be a noose in $\tilde{\Gamma}$ such that:

$$N'\cap\tilde{\Gamma}=\{v_1^{\text{up}\prime},\ldots,v_{k-2}^{\text{up}\prime}=v_1^{\text{right}\prime},\ldots,v_{k-2}^{\text{right}\prime}=v_{k-2}^{\text{down}\prime},\ldots,v_1^{\text{down}\prime}=v_{k-2}^{\text{left}\prime},\ldots,v_1^{\text{left}\prime}=v_1^{\text{up}\prime}\}.$$

Notice that all vertices of $N\cap\tilde{\Gamma}$, along with the edges of $\tilde{\Gamma}$ that have at least one endpoint in $N\cap\tilde{\Gamma}$ are in one, say D^*, of the open disks N' defines. We define $\Gamma'=\tilde{\Gamma}\setminus D^*$. We also define $\mathcal{P}^{\leftrightarrow\prime}=\{P_1^{\leftrightarrow\prime},\ldots,P_{k-2}^{\leftrightarrow\prime}\}$ and $\mathcal{P}^{\updownarrow\prime}=\{P_1^{\updownarrow\prime},\ldots,P_{k-2}^{\updownarrow\prime}\}$ such that each path $P_i^{\leftrightarrow\prime}/P_i^{\updownarrow\prime}$ is obtained by $\tilde{P}_{i+1}^{\leftrightarrow}/\tilde{P}_{i+1}^{\updownarrow}$ after removing its two endpoints. Notice that Γ' is the union of the paths in $\mathcal{P}^{\leftrightarrow\prime}$ and the paths in $\mathcal{P}^{\updownarrow\prime}$. Also the path collections $\mathcal{P}^{\leftrightarrow\prime}$, $\mathcal{P}^{\updownarrow\prime}$ and the vertices in $N'\cap\Gamma'$ satisfy conditions *(i)* and *(ii)* of Lemma 3.5.25. Thus $\Gamma'_{k-2}(T)$. We now consider the graph Λ_{k-2} and we set up a partial function $\eta':V(\Lambda_{k-2})\to\Gamma'$ such that:

$\eta'\big((1,1)\big)=v_1^{\text{up}\prime},\ldots,\eta'\big((1,k)\big)=v_k^{\text{up}\prime}$,
$\eta'\big((1,k)\big)=v_1^{\text{right}\prime},\ldots,\eta'\big((k,k)\big)=v_k^{\text{right}\prime}$,
$\eta'\big((k,k)\big)=v_k^{\text{down}\prime},\ldots,\eta'\big((k,1)\big)=v_1^{\text{down}\prime}$, and
$\eta'\big((k,1)\big)=v_k^{\text{left}\prime},\ldots,\eta'\big((1,1)\big)=v_1^{\text{left}\prime}$.

Since $\Gamma'_{k-2}(T)$, by the induction hypothesis, there exist a function $\alpha':V(\Lambda_{k-2})\to V(\Gamma')$ such that: $\eta'\subseteq\alpha'$ and Λ_{k-2} is an α'-rooted minor of Γ'.

This means that Γ' can be transformed, by applying a series of $\alpha'(V(\Lambda_{k-2}))$-maintaining edge contractions and edge removals, to a new graph Γ^* such that Λ_{k-2} is isomorphic to Γ^* via some isomorphism $\rho:V(\Lambda_{k-2})\to V(\Gamma^*)$ where $\alpha'\subseteq\rho$.

Consider the graph that we obtain from Λ_k if:

(i) we remove the $4k-4$ vertices $\{(1,i),(i,1),(i,k),(k,i)|1\le i\le k\}$;

(ii) we remove the edges $\{\{(1,i),(1,i+1)\}|1\le i\le k-1\ \}\cup\{\{(i,k),(i+1,k)\}|1\le i\le k-1\ \}\cup\{\{(k,i),(k,i+1)\}|1\le i\le k-1\ \}\cup\{\{(i,1),(i+1,1)\}|1\le i\le k-1\ \}$

(iii) we remove the edges $\{\{(1,i),(2,i)\}|2\le i\le k-1\}\cup\{\{(i,k),(i,k-1)\}|2\le i\le k-1\}\cup\{\{(k,i),(k-1,i)\}|2\le i\le k-1\}\cup\{\{(i,1),(i,2)\}|2\le i\le k-1\}$.

We call $\tilde{\Lambda}_{k-2}$ this graph. As it is indicated by Definition 3.5.6 the graph $\tilde{\Lambda}_{k-2}$ it is isomorphic to the graph Λ_{k-2}. Let $\beta:V(\tilde{\Lambda}_{k-2})\to V(\Lambda_{k-2})$ be an isomorphism which witness that $\tilde{\Lambda}_{k-2}$ is isomorphic to Λ_{k-2}. Then the function $\rho\circ\beta:V(\tilde{\Lambda}_{k-2})\to V(\Gamma^*)$ witnesses that $\tilde{\Lambda}_{k-2}$ is isomorphic to Γ^*.

Let $\tilde{\Gamma}''$ be the graph that we obtain if we apply on $\tilde{\Gamma}$ the same sequence of operations that transforms Γ' to Γ^*. and consider the function $\alpha:=\eta\cup\rho\circ\beta$. Then

(i) $\eta\subseteq\alpha$;

(ii) α is an isomorphism between Λ_k and the graph $\tilde{\Gamma}''$;

(iii) the graph $\tilde{\Gamma}''$ is the result of a sequence of $\alpha(V(\Lambda_k))$-maintaining contractions and edge removals on Γ.

And thus we have $\Pi_k(T)$.

Induction Conclusion: Claim 3.5.26 holds. □

The proof of Claim 3.5.26 completes the proof of Lemma 3.5.25. □

Lemma 3.5.27. There exists an algorithm, that given as an input a simple 2-connected n-vertex plane graph Γ and an integer $k \geq 2$, returns as an output either a minor model of the $(k \times k)$-grid in Γ, either a sphere-cut decomposition of Γ of width at most $4k - 4$. Moreover, this algorithm runs in $O(n^3)$ steps.

Proof. The lemma follows easily when $k = 2$. To see this observe that any 2-connected graph that is not a triangle contains a cycle of at least 4 vertices and therefore a (2×2)-grid-minor and that is it is trivial to construct a sphere-cut branch-decomposition of a triangle of width $2 \leq 4 \cdot 2 - 4$.

In what remains, we examine the non-trivial case where $k \geq 3$. As Γ is 2-connected and simple, the boundary of each of its faces is a cycle with at least 3 vertices.

The algorithm starts by picking an arbitrary edge e_0 of Γ and constructs a partial sphere-cut decomposition $B_0 = (T_0, \tau_0)$ so that T_0 has 2 leaves l, l', $\tau_0^{-1}(l) = \{e_0\}$, and $\tau_0^{-1}(l') = E(\Gamma) \setminus \{e_0\}$. Notice that the 2-connectivity of Γ implies that B_0 has width $2 \leq 4k - 4$.

Algorithm 1: Grid minor or small branch-width.

Result: Either a sphere-cut branch-decomposition of Γ of width $\leq 4k - 4$,
either a minor model of Λ_k in Γ

```
initialization ;                                   // Let E(Γ) = {e_0, ..., e_{m-1}}
T ← ({l, l'}, {{l, l'}}), τ ← {(e_0, l), (e_1, l'), ... (e_{m-1}, l')};
B ← (T, τ);
Λ_k ← False, σ ← ∅ ;          // Λ_k is a boolean variable which is true iff Λ_k ≤_m Γ.
Incomplete-SCBD ← True ;   // Incomplete-SCBD is a boolean variable which is true
  iff B is not a complete branch-decomposition of Γ.
while (Incomplete-SCBD = True) and(Λ_k = False) do
  | (Incomplete-SCBD, Λ_k, B, σ) ← Procedure(B) ; // "Procedure" is the procedure
  |   described in the course of the proof.
end
if Λ_k = True, then
  | return σ;
else
  | return B;
end
```

The rest of the proof is dedicated to the presentation of the procedure that receives an incomplete partial sphere-cut decomposition $B = (T, \tau)$ of Γ of width $\leq 4k - 4$ and outputs either the model of a $(k \times k)$-grid minor of Γ, either a partial sphere-cut decomposition $B' = (T', \tau')$ of Γ of width $\leq 4k - 4$ and where such that $|L(T')| > |L(T)|$. Clearly, if we iteratively apply this procedure, starting from B_0, until $|L(T')| = |E(\Gamma)|$ the output is one of the two possible outcomes of the lemma.

We now proceed with the description of the above procedure. Let ω be the function certifying that the input $B = (T, \tau)$ is a partial sphere cut decomposition. Let e be a loaded edge of B, incident to some loaded leaf l_1 of T. Let also $N_e = \omega(e)$. As Γ is 2-connected, we have that $|N_e| \geq 2$. We define the *interior* of N_e as the connected component of $\mathbb{R}^2 \setminus N_e$ that contains all the edges in $\tau^{-1}(l_1)$, and we denote it by $\mathbf{in}(N_e)$. Clearly, $\mathbf{in}(N_e)$ is an open disk. Let $\mathcal{I} = \{I_1, \dots, I_{|N_e|}\}$ be the collection of arcs of Γ that constituting the connected components of $N_e \setminus \Gamma$. We distinguish two cases **Case 1**, **Case 2**. Each of them has two subcases. In the one subcase of **Case 2** the output is a minor model of the $(k \times k)$-grid in Γ. In the other subcases the output is two nooses N_1 and N_2 such that:

A. (N_1, N_2, N_e) is a proper Θ-triple of Γ that induces some partition $\{E_1, E_2, E_3\}$ of $E(\Gamma)$ where $E_3 = E(\Gamma) \setminus \tau^{-1}(e)$. and

B. $|N_1|, |N_2| \leq 4k - 4$

Case 1. $|N_e| < 4k - 4$. Let I be some of the arcs in $\mathcal{I}$ and let f be the face of Γ that contains this arc. Let also x and y be the endpoints of I and let $e_x = \{x, x'\}$ be the edge in the boundary of f that is inside in $\mathbf{in}(N_e)$. We examine two subcases:

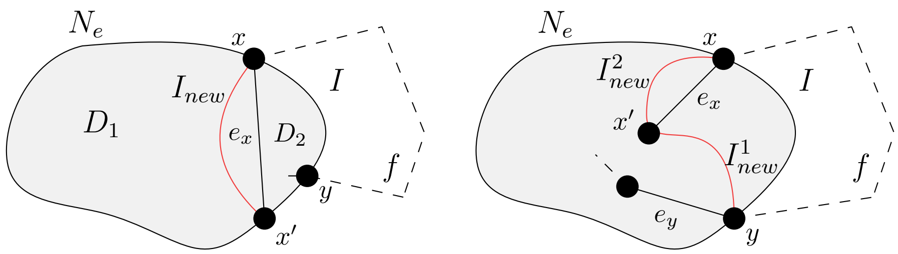

Figure 3.5.11: Deducing the nooses N_1, N_2 in subcases $1.a$, $1.b$ of the proof of Lemma 3.5.27.

Subcase 1.a. $x' \in V(N_e)$, i.e. x' is a vertex of $\mathbf{mid}(e)$. Consider an arc I_{new} of Γ between x and x' and inside in $\mathbf{in}(N_e)$ such that one, say D_1, of the two connected components of $\mathbf{in}(N_e) \setminus I_{\text{new}}$ contains the edge e_x and the other, say D_2, contains some edge in $\tau^{-1}(l_1) \setminus \{e_x\}$. Let E_1 and E_2 be the sets of edges inside D_1 and D_2 respectively. We observe that E_1 and E_2 form a partition of $\tau^{-1}(e)$. The algorithm outputs the boundaries N_1 and N_2 of D_1 and D_2 respectively. By construction, the triple (N_1, N_2, N_e) satisfies condition **A**. Observe that $|N_1| \geq 2$, $|N_2| \geq 2$ and, from Observation 3.5.19, $|N_1| + |N_2| = |N_e| + 2$. Therefore, $|N_1|, |N_2| \leq |N_e| \leq 4k - 4$ and condition **B** holds as well.

Subcase 1.b. $x' \notin V(N_e)$, therefore $x' \in \mathbf{in}(N_e)$. Let e_y be the edge on the boundary of f that has y as endpoint and is inside $\mathbf{in}(N_e)$. We can assume that e_x and e_y are different edges as, otherwise, $y = x'$ and the previous subcase applies.
Consider an arc I^1_{new} between y and x' and an arc I^2_{new} between x' and x in a way that if N_1 is the union of I, I^1_{new}, I^2_{new}, and the points x, y, and x', then the edges e_x and e_y are separated by N_1. We now partition the edges of $\tau^{-1}(e)$ to two sets E_1 and E_2 where E_1 contains the edges of $\tau^{-1}(e)$ that are inside one of the two open disks bounded by N_1 and E_2 contains the rest. Let also N_2 be the noose occurring if we remove the arc I from the union of N_e and N_1. By construction, the triple

(N_1, N_2, N_e) satisfies condition **A**. We also observe that $|N_1| = 3$ and, from Observation 3.5.19, $|N_1| + |N_2| = |N_e| + 2$. We conclude that $|N_1|, |N_2| \leq |N_e| + 1 \leq 4k - 4$ and Condition **B** holds as well.

Case 2. $|N_e| = 4k - 4$. We name the vertices in $V(N_e)$ by

$$v_1^{\text{up}}, \ldots, v_k^{\text{up}} = v_1^{\text{right}}, \ldots, v_k^{\text{right}} = v_k^{\text{down}}, \ldots, v_1^{\text{down}} = v_k^{\text{left}}, \ldots, v_1^{\text{left}} = v_1^{\text{up}},$$

following their clock-wise cyclic ordering on N_e. We also set $V^{\text{up}} = \{v_1^{\text{up}}, \ldots, v_k^{\text{up}}\}$, $V^{\text{right}} = \{v_1^{\text{right}}, \ldots, v_k^{\text{right}}\}$, $V^{\text{down}} = \{v_1^{\text{down}}, \ldots, v_k^{\text{down}}\}$, and $V^{\text{left}} = \{v_1^{\text{left}}, \ldots, v_k^{\text{left}}\}$. Let $\Gamma^{\leftrightarrow}$ be the graph obtained if we add in Γ the vertices z^{left} and z^{right} and connect all vertices in V^{left} with z^{left} and all vertices in V^{right} with z^{right}. Also let $\Gamma^{\updownarrow}$ be the graph obtained if we add in Γ the vertices z^{up} and z^{down} and connect all vertices in V^{up} with z^{up} and all vertices in V^{down} with z^{down}.

Let $S^{\leftrightarrow}$ be a minimum $(z^{\text{left}}, z^{\text{right}})$-separator of $\Gamma^{\leftrightarrow}$ and let $S^{\updownarrow}$ be a minimum $(z^{\text{up}}, z^{\text{down}})$-separator of $\Gamma^{\updownarrow}$. Because of the 2-connectivity of Γ, we have that both $|S^{\leftrightarrow}|$ and $|S^{\updownarrow}|$ have at least 2 vertices. We examine two subcases:

Subcase 2.a. $|S^{\leftrightarrow}| \geq k$ and $|S^{\updownarrow}| \geq k$. From Menger's Theorem 1.2.73 there is a collection $\mathcal{R}^{\leftrightarrow}$ of k internally disjoint paths from z^{left} to z^{right} in $\Gamma^{\leftrightarrow}$ and a collection $\mathcal{R}^{\updownarrow}$ of k internally disjoint paths from z^{up} to z^{down} in $\Gamma^{\updownarrow}$. Let $\mathcal{P}^{\leftrightarrow} = \{P_1^{\leftrightarrow}, \ldots, P_k^{\leftrightarrow}\}$ be the paths obtained from the paths in $\mathcal{R}^{\leftrightarrow}$ after removing from them the vertices z^{left} to z^{right} and let $\mathcal{P}^{\updownarrow} = \{P_1^{\updownarrow}, \ldots, P_k^{\updownarrow}\}$ be the paths obtained from the paths in $\mathcal{R}^{\updownarrow}$ after removing from them the vertices z^{up} to z^{up}. We consider the subgraph Γ' of Γ obtained by the union of the paths in $\mathcal{R}^{\leftrightarrow}$ and the paths in $\mathcal{R}^{\updownarrow}$. Notice that the paths in $\mathcal{R}^{\leftrightarrow}$ and $\mathcal{R}^{\updownarrow}$ meet the specifications of Lemma 3.5.25, therefore Γ', and therefore Γ as well, contains a $(k \times k)$-grid as a minor.

Subcase 2.b. $|S^{\leftrightarrow}| < k$ or $|S^{\updownarrow}| < k$. Without loss of generality we assume that $|S^{\leftrightarrow}| < k$. From Lemma 3.5.20, there is a noose N in $\Gamma^{\leftrightarrow}$ such that $V(N) = S^{\leftrightarrow}$. Notice that, as Γ is 2-connected, it holds that $|S^{\leftrightarrow}| = |N| \geq 2$ and this sub-case is applied only when $k \geq 3$.

Let f_{out} be the unique face of $\Gamma^{\leftrightarrow}$ that contains both z^{left} and z^{right} in its boundary. We insist that the set $I = N \setminus f_{\text{out}}$ is an arc of $\Gamma^{\leftrightarrow}$ that is a subset of $\mathbf{in}(N_e) \cup V(N_e)$, as we can always update N with an equivalent one that has this property. By definition, I is also an arc of Γ. Observe that I contains a sub-arc I' so that the one endpoint of I' is a vertex of V^{up}, the other is a vertex of V^{down}, and I' does not contain any other vertex in $V^{\text{up}} \cup V^{\text{down}}$ except from its endpoints. Observe also that $|I'| \leq |I| = |N| = |S^{\leftrightarrow}| < k$ and keep in mind that $I' \subseteq \mathbf{in}(N_e) \cup V(N_e)$.

Claim 3.5.28. *There is an arc I'' in $\mathbf{in}(N_e)$ with endpoints $a, b \in V(N_e)$, such that $I'' \cap N_e = \emptyset$. Moreover, if $I^{(1)}$ and $I^{(2)}$ are the two arcs in N_e that have a and b as endpoints, then $|I''| < |I^{(i)}|, i \in \{1, 2\}$.*

Proof of claim. We examine first the case where I' contains a sub-arc I'' with one endpoint, say a, in V^{left}, the other, say b, in V^{right}, and no other vertex in N_e. Notice that every arc $I^{(i)}, i \in \{1, 2\}$ in N_e that has one of its endpoints in V^{left} and the other in V^{right} contains either all the vertices of V^{up} or all the vertices of V^{down}. This implies that $k \leq |I^{(i)}|, i \in \{1, 2\}$. Therefore, $|I''| < |I'| \leq k \leq |I^{(i)}|, i \in \{1, 2\}$, and the claim follows.

The case where I' contains a sub arc I'' with one endpoint in V^{up}, the other in V^{down} and no other vertex in N_e is the same as the above case if we replace V^{left} with V^{up} and V^{right} with V^{down}.

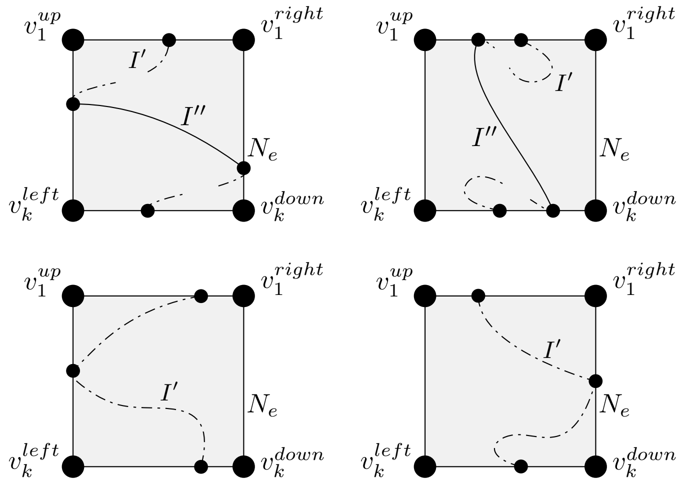

Figure 3.5.12: Illustration of the possible cases that we face while trying to deduce the arc I'' in the proof of Claim 3.5.28

In the remaining case, we have that either I' intersects $V^{\text{left}} \setminus \{v_1^{\text{up}}, v_1^{\text{down}}\}$ or some vertex of $V^{\text{right}} \setminus \{v_k^{\text{up}}, v_k^{\text{down}}\}$ but not from both. W.l.o.g., we assume that I' contains some vertex from $V^{\text{left}} \setminus \{v_1^{\text{up}}, v_1^{\text{down}}\}$ but none from $V^{\text{right}} \setminus \{v_k^{\text{up}}, v_k^{\text{down}}\}$. Let I^* be the arc in N_e with the same endpoints as I' and with some endpoint in $V^{\text{left}} \setminus \{v_1^{\text{up}}, v_1^{\text{down}}\}$; notice that I^* is well defined as $V^{\text{left}} \setminus \{v_1^{\text{up}}, v_1^{\text{down}}\}$ is non-empty (because $k \geq 3$). Clearly, $V(I^*) \supseteq V^{\text{left}}$. This implies that $|I^*| \geq k$, therefore $|I'| < |I^*|$. Consider now the set $(I' \cup I^*) \setminus (I' \cap I^*)$ and observe that its connected components are arcs. Notice also that these arcs can be enumerated as $I'_1, \ldots, I'_\rho$ and $I^*_1, \ldots, I_\rho*$ such that each I'_i is a sub-arc of I', each I^*_i is a sub-arc of I^*, and, for $i = \{1, \ldots, \rho\}$, either $I'_i = I^*_\rho$ or I'_i and I^*_i are disjoint and with common endpoints. Let $R = \{i \in \{1, \ldots, \rho\} | I'_i \cap I^*_i = \emptyset\}$ and observe that $|I'| < |I^*|$ implies that $\sum_{i \in R} |I'_i| < \sum_{i \in R} |I^*_i|$. This in turn implies that, for some $i \in R$, $|I'_i| < |I^*_i|$. We set $I'' = I'_i$ and $I^{(1)} = I^*_i$ and keep in mind that $|I''| < |I^{(1)}| \leq k$. Let a and b be the endpoints of I'' and notice that I'' does not have any point in N_e, apart from a and b. As $I^{(1)} \subseteq N$, I_{left} is one of the two arcs in N that have a and b as endpoints. Clearly, the other arc in N with the same property is $I^{(2)} = (N \setminus I_{\text{left}}) \setminus \{a\} \setminus \{b\}$. As $I^{(2)}$ intersects or has as endpoints all the vertices in V^{right}, we have that $|I_{\text{right}}| \geq k$. We conclude that $|I''| < |I^{(1)}| \leq k \leq |I^{(2)}|$ and this completes the proof of the claim. □

From the above claim, there is an arc I'' such that $I'' \subseteq \mathbf{in}(N_e)$, the endpoints a and b of I'' are in $V(N_e)$, and if I_1 and I_2 are the two arcs in N_e with endpoints a and b, then $|I''| \leq |I_1|$ and $|I''| \leq |I_2|$. Notice also that $\mathbf{in}(N_e) \setminus I''$ has two connected components D_1, D_2 that are open disks. For $i \in \{1, 2\}$, let E_i be the edges inside D_i. By construction, none of E_1, E_2 is an empty set, therefore they form a partition of $\tau^{-1}(e)$. Let also N_1 and N_2 be the boundaries of D_1 and D_2 such that $N_1 = I_1 \cup I''$ and $N_2 = I_2 \cup I''$. Clearly, the triple (N_1, N_2, N_e) satisfies condition **A**. As, for every $i \in \{1, 2\}$, $|N_i| = |I_i| + |I''| - 2 \leq |I_i| + |I_{3-i}| - 2 = |N_e| \leq 4k - 4$, Condition **B** holds as well.

In each of the sub-cases *1.a*, *1.b*, and *2.b*, we obtained two nooses N_1 and N_2 where conditions **A** and **B** holds. Let $\{E_1, E_2, E_3\}$ be the partition of $E(\Gamma)$ induced by the Θ-triple (N_1, N_2, N_e).

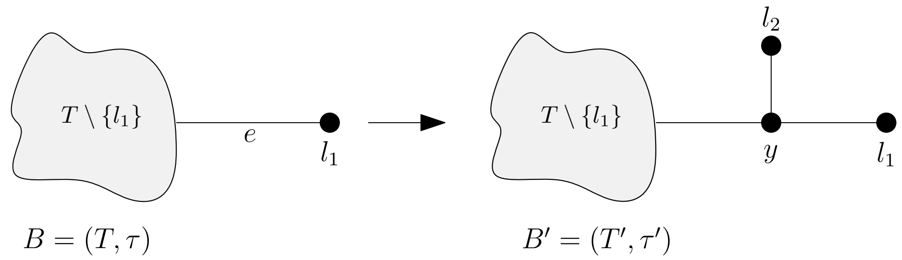

Figure 3.5.13: Increasing the valency of the incomplete partial sphere-cut decomposition $B = (T, \tau)$ of Γ in cases 1a, 1b, 2b of the proof of Lemma 3.5.27

We now define $B' = (T', \tau')$ by modifying $B = (T, \tau)$ as follows: let T' be the tree obtained from T if we subdivide the edge e and make the subdivision vertex y adjacent with a new leaf l_2. We then define $\tau' : E(G) \to L(T')$ such that $\tau'(l_i) = E_i, i \in \{1, 2\}$, and $\tau'(l) = \tau(l)$, for every $l \in L(T') \setminus \{l_1, l_2\}$. We also define the function ω' such that $\omega'(l_i) = N_i, i \in \{1, 2\}$, and $\omega'(l) = \omega(l)$ for every $l \in L(T') \setminus \{l_1, l_2\}$. This definition, together with Condition **B**, implies that $B' = (T', \tau')$ is a partial sphere-cut decomposition of Γ of width $\leq 4k-4$ where $|L(T')| > |L(T)|$, as required in the specifications of the main procedure of the algorithm.

Notice that the main procedure is applied $O(n)$ times and its running time is dominated by the computation of the k-disjoint paths or the separator of size $< k$ in Subcase *2.b*, which, in planar graphs, can be done in $O(n^2)$ steps. The claimed overall running time follows. □

Proof of Theorem 3.5.24. Immediate from Theorem 3.3.39 and Lemma 3.5.27. □

3.5.4 Well-quasi-ordering planar graphs

Proof of Theorem 3.5.2. Let Γ be an arbitrary but fixed planar graph and let us denote by $\mathcal{P}$ the set of all planar graphs with no minor isomorphic to Γ.

Since from Theorem 3.5.9 there exist an integer k such that the graph Γ is isomorphic to a minor of the $(k \times k)$-grid, it follows from the transitivity of the minor relation that no graph in $\mathcal{P}$ has a minor isomorphic to the $(k \times k)$-grid.

By the Excluded Grid Theorem for planar graphs (Theorem 3.5.24) every planar graph with branch-width greater than $4k - 4$ contains the $(k \times k)$-grid as a minor. Thus $(\forall G \in \mathcal{P})[\mathbf{bw}(G) \leq 4k - 4]$.

Hence by Theorem 3.4.2 it follows that the set $\mathcal{P}$ is well-quasi-ordered by the minor relation. Since Γ was an arbitrary planar graph the proof is complete. □

Proof of Theorem 3.5.1. Let $G_1, G_2, \ldots$ be an arbitrary but fixed infinite sequence of planar graphs, such that the graph G_1. If there exists a positive integer $j > 1$ such that G_1 is isomorphic to a minor of G_j, we are done. If for every positive integer $j > 2$ the graph G_j has no minor isomorphic to the planar graph G_1 then by Theorem 3.5.2 the set of graphs $\{G_j | j > 1\}$ is well-quasi-ordered by the minor relation (as a subset of the set of all planar graphs without a minor isomorphic to G_1). Thus the sequence $(G_j)_{j \geq 2}$ is an infinite sequence of a well-quasi-ordered set, and hence must contain at least one good pair of graphs, that is, there exist i, j with $j > i$ such that that G_i is isomorphic to a minor of G_j. Since $G_1, G_2, \ldots$ was an arbitrary infinite sequence of planar graphs, our proof is complete. □

3.6 Graphs which exclude a fixed planar graph as a minor

The purpose of this section is to present the proof of the following special case of Robertson and Seymour's theorem.

Theorem 3.6.1 (Robertson and Seymour [109])**.** If $G_1, G_2, \ldots$ is any infinite sequence of graphs, such that G_1 is planar. Then there exist i, j with $j > i \geq 1$ such that G_i is isomorphic to a minor of G_j.

The main ingredient that we will need is the following:

Theorem 3.6.2 (Robertson and Seymour [108])**.** For any planar graph H, the set of all graphs with no minor isomorphic to H is well-quasi-ordered by the minor relation.

Since -by Theorem 3.5.9- every planar graph is minor of a large enough grid, the transitivity of the minor relation on graphs implies that when a graph, say G, does not contain a fixed planar graph as a minor then there is a positive integer k, such that G has not a minor isomorphic to the $(k \times k)$-grid. Hence we can deduce informations about the "rough" structure of graphs which exclude a fixed planar graph as a minor, by studying the case in which the excluded minor is a grid.

This case is studied on the Excluded Grid Theorem (Theorem 3.6.3) which is one of the two basic ingredients for the proof of Theorem 3.6.2. Informally, the Excluded Grid Theorem (Theorem 3.6.3) states that if a graph has large tree-width (or, similarly, -due to Theorem 3.3.33-, branch-width) then it contains a large grid as a minor. Thus if a graph excludes a grid as a minor then it has bounded tree-width and thus bounded branch-width.

Hence, for any planar graph H, the set of all graphs with no minor isomorphic to H is a set of graphs with bounded branch-width. Here comes the second ingredient which is Theorem 3.4.2 which states that any set of graphs with bounded branch-width is well-quasi-ordered by the minor relation and has already been proved in Subsection 3.4.3.

3.6.1 The excluded grid theorem

The Excluded Grid Theorem (also called Grid Theorem and Grid Minor Theorem), is the following:

Theorem 3.6.3 (Excluded Grid Theorem, Robertson and Seymour [108])**.** For each positive integer k, there is an integer $g(k)$ such that every graph with tree-width at least $g(k)$ has an $(k \times k)$-grid minor.

It was first proved by Robertson and Seymour [108, Theorem 7.3] and the first short proof was given by Diestel, Jensen, Gorbunov, and Thomassen [34]. In this subsection we prove the Excluded Grid Theorem. Actually due to the close relationship of the graph invariants branch-width and tree-width (Theorem 3.3.33), which implies the equivalent of Theorem 3.6.3 with Theorem 3.6.4 and the fact that the proof of the Excluded Grid Theorem is slightly easier in the branch-width version, we are working with branch-width and we prove Theorem 3.6.4. The proof is based on [34], and the presentation follows [102].

Theorem 3.6.4 (Excluded Grid Theorem (branch-width version))**.** For each positive integer k, there is an integer $f(k)$ such that every graph with branch-width at least $f(k)$ has an $(k \times k)$-grid minor.

From now and on when we refer to the Excluded Grid Theorem -if it is not stated otherwise- we refer to its branch-width version, that is, to Theorem 3.6.4.

Let us now outline the course of the proof of the Excluded Grid Theorem. Our basic ingredients are two lemmas, Lemma 3.6.13 and Lemma 3.6.19.

The first lemma states that given three positive integers, say k, s and t, and provided a specific structure (see Definition 3.6.5) -which is depended on k, s and t- on a graph we can find on this graph either a $(k \times k)$-grid minor or a set of $s + t$ disjoint paths which meets some requirements.

In the second lemma we prove that given a graph with "sufficiently" large branch-width we can find a specific structure -which depends on the branch-width of the graph- in it.

Then, given an integer k, the proof of Excluded Grid Theorem lies in choosing the right integer $f(k)$ which is such that if a graph, say G, has branch-width at least $f(k)$ then the structure which exists by our second lemma on G is appropriate for applying a generalization (Corollary 3.6.18) of the first lemma and find -either directly or by making use of the disjoint paths (the existence of which is guaranteed by Corollary 3.6.18)- the desired $(k \times k)$-grid minor on G.

In order to state the two lemmas that we need for the proof of the Excluded Grid Theorem we first need to set up some notation and to state some definitions.

For the rest of this section when it is clear from the context in which graph we refer, we do not use subscript for the connectivity function of this graph. Recall from Observation 3.3.36 that given a graph G and two sets of vertices $A, B \subseteq V(G)$, the number $\gamma(A, B)$ equals with the maximum size of a set of disjoint (A, B)-paths, where γ is the connectivity function of the graph G.

Definition 3.6.5 ((p, q)-path-system)**.** A sextuple $(G, A, B, X, Y, \mathcal{P})$ consisting of a graph G, four disjoint subsets A, B, X, Y of $V(G)$, and a set $\mathcal{P}$ of disjoint (A, B)-paths will be said to be a (p, q)-*path-system* if $|\mathcal{P}| \geq p$ and there exist q disjoint (X, Y)-paths in G.

Notation 3.6.6. Let G be a graph, let also W, X and Y be subsets of $V(G)$ such that W separates X from Y. We denote by E_W the set of edges of G which contains those $e \in E(G)$ for which there is a path P in G, containing e, such that P has its one endpoint at $X \setminus W$ and $P \cap W$ consists of at most one end of P.

Observation 3.6.7. If G is a graph and W, X, Y are subsets of $V(G)$ such that W separates X from Y, then $\Gamma(E_W) \subseteq W$ and $\Gamma(E_W) \cup (W \cap (X \cup Y))$ separates X from Y in G.

Notation 3.6.8. Let G be a graph, $X, Y \subseteq V(G)$ and $E' \subseteq E(G)$. We denote by $\Gamma_{(X,Y)}(E')$ the union of $\Gamma(E')$ with all $x \in X$ which are incident with an edge of $E(G) \setminus E'$ and with all $y \in Y$ which are incident with an edge in E'.

Observation 3.6.9. Let G be a graph, $X, Y \subseteq V(G)$ and $E' \subseteq E(G)$. The set $\Gamma_{(X,Y)}(E')$ is not in general equal with the set $\Gamma_{(Y,X)}(E')$, they differ as to which elements of X and Y are included.

Proposition 3.6.10. Let G be a graph, $X, Y \subseteq V(G)$ and $E' \subseteq E(G)$, then the set $\Gamma_{(X,Y)}(E')$ separates X from Y in G.

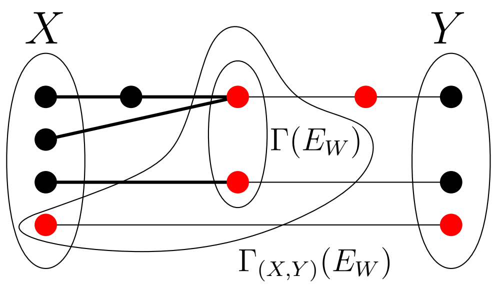

Figure 3.6.1: Illustration of Notations 3.6.6, 3.6.8. With fatter lines are drawn the edges of the set E_W and the vertices of W are colored with red.

Proof. Let P be an arbitrary but fixed (X, Y)-path, we distinguish the following cases:

Case 1: The path P consists in exactly one edge, say $e = \{u, v\}$ with $u \in X, v \in Y$.
Sub-case 1a: $e \in E'$.
In that case $v \in \Gamma_{(X,Y)}(E')$
Sub-case 1b: $e \notin E'$.
In that case $u \in \Gamma_{(X,Y)}(E')$

Case 2: The path $P = (u, \ldots, v)$ consists in at least 2 edges.
Sub-case 2a: All the edges of P are in E'.
In that case $v \in \Gamma_{(X,Y)}(E')$.
Sub-case 2b: None of the edges of P is in E'.
In that case $u \in \Gamma_{(X,Y)}(E')$.
Sub-case 2c: P has at least one edge in E' and one edge in $E \setminus E'$.
In that case P contains at least one vertex of $\Gamma(E')$ and thus $\Gamma_{(X,Y)}(E')$ contains at least one vertex of P.

In any case, one vertex of P is in $\Gamma_{(X,Y)}(E')$. Since P was an arbitrary (X, Y)-path it follows that the set of vertices $\Gamma_{(X,Y)}(E')$ is a (X, Y)-separator. □

It is easy to see that the following holds.

Proposition 3.6.11. If G is a graph and W, X, Y are subsets of $V(G)$ such that W separates X from Y, then $\Gamma_{(X,Y)}(E_W) \subseteq W$.

Corollary 3.6.12. Let G be a graph, $X, Y \subseteq V(G)$, then all minimal sets separating X from Y are of the form $\Gamma_{(X,Y)}(E')$ for some $E' \subseteq E(G)$.

Lemma 3.6.13. Let k, s and t be given positive integers. Then there exist positive integers p and q such that, for any (p, q)-path-system $(G, A, B, X, Y, \mathcal{P})$, either G has an $(k \times k)$-grid minor, or there is a subset $\mathcal{P}'$ of $\mathcal{P}$ of size s and a set $\mathcal{Q}$ of (X, Y)-paths of size t such that $\mathcal{P}' \cup \mathcal{Q}$ is a set of disjoint paths.

Proof. Given k, s and t, let r and $q \geq 2s + t$ be large enough so that the bipartite Ramsey result holds, i.e. any coloring of the edges of the complete bipartite graph $K_{2r,q}$ with two colors (red and blue) either has a red $K_{s,t}$ subgraph or a blue K_{2k^{2k+1},k^2} subgraph. Set $p := s + (2r+2)q$ and let $(G, A, B, X, Y, \mathcal{P})$ be a (p, q)-path-system.

We begin with some simple observations. If G has an edge e not in any path in $\mathcal{P}$, so that there are at least q disjoint (X, Y)-paths in $G \setminus e$, then we may freely delete e and continue with $G \setminus e$ in place of G. Similarly, consider an edge e in a path $P \in \mathcal{P}$ such that P has length at least 2. If G/e has at least q disjoint (X, Y)-paths, then we proceed with G/e in place of G.

Thus, we may assume that any edge not in any path in $\mathcal{P}$ is in every collection of q disjoint (X, Y)-paths in G, and likewise, every edge in a path in $\mathcal{P}$, as long as it is not the only edge in that path, joins vertices in a (X, Y)-separation of size q.

If there are at least s paths of length 1 in $\mathcal{P}$, then we may use these paths for $\mathcal{P}'$. As long as $q \geq t + 2s$, any set of q disjoint (X, Y)-paths contains at least t paths that are all disjoint from the paths in $\mathcal{P}'$. So we may suppose that $\mathcal{P}$ contains less than s paths of length 1. Furthermore, we may assume that $|X| = q = |Y|$. Notice that $\Gamma_{(X,Y)}(\emptyset) = X$ and $\Gamma_{(X,Y)}(E(G)) = Y$.

Our next step is to find an appropriate set of cuts. Let $\mathcal{P}_1$ denote the subset of $\mathcal{P}$ consisting of all the paths of length at least 2.

Claim 3.6.14. *We shall find a sequence of* $2r + 2$ *sets* $E_0 = \emptyset \subseteq E_1 \subseteq E_2 \subseteq \ldots \subseteq E_{2r} \subseteq E(G) = E_{2r+1}$ *of edges such that, for each* i, $|\Gamma_{(X,Y)}(E_i)| = q$, *and each* $\Gamma_{(X,Y)}(E_i)$ *contains the endpoints of an edge of some path* $P_i \in \mathcal{P}_1$ *such that* $V(P_i)$ *is disjoint from all the other* $\Gamma_{X,Y}(E_j)$.

Proof of Claim 3.6.14. Suppose that, for some $m \geq 0$, we have found sets of edges $E_0 = \emptyset, E_1, \ldots, E_m, E_{m+1} = E(G)$ and paths $P_1, P_2, \ldots, P_m$ so that E_i are nested, for each i, $|\Gamma_{(X,Y)}(E_i)| = q$, and each P_j has an edge whose endpoints are in $\Gamma_{(X,Y)}(E_j)$ but not in any other $\Gamma_{(X,Y)}(E_k)$. It is easy to see that, as long as $|\mathcal{P}_1| \geq (m+2)q + 1$, there is a path $P \in \mathcal{P}_1$ that is disjoint from all the sets $\Gamma_{(X,Y)}(E_i)$ for $i \in \{0, 1, \ldots, m+1\}$.

Let i be least such that $E(P) \subseteq E_i$. Then, because P is disjoint from all the $\Gamma_{(X,Y)}(E_j)$, $E(P) \subseteq E_i \setminus E_{i-1}$.

Contract any edge e of P. From the basic reductions, there do not exist q disjoint (X, Y)-paths in G. This easily implies that there are not q disjoint $(\Gamma_{(X,Y)}(E_{i-1}), \Gamma_{(X,Y)}(E_i))$-paths. Thus, there is an E' such that $E_{i-1} \subseteq E' \subseteq E_i$, the endpoints of e are in $\Gamma_{(X,Y)}(E')$, and $|\Gamma_{(X,Y)}(E')| = q$. We proceed to the next iteration with the sequence $E_0, E_1, \ldots, E_{i-1}, E', E_i, \ldots, E_m, E_{m+1}$ and the paths $P_1, \ldots, P_{i-1}, P, P_i, \ldots, P_m$.

Since $p \geq s - 1 + (2r+2)q + 1$, we now have the desired sequences $E_0, E_1, \ldots, E_{2r}, E_{2r+1}$ and $P_1, P_2, \ldots, P_{2r}$. □

Next fix any set $\mathcal{Q}$ of q disjoint (X, Y)-paths and let $\mathcal{P}' = \{P_1, P_2, \ldots, P_{2r}\}$. By the bipartite Ramsey result, either there are s paths in $\mathcal{P}'$ and t paths in $\mathcal{Q}$ that are all disjoint (in which case we are done) or there are a subset $\mathcal{P}''$ of $\mathcal{P}'$ of size $2k^{2k+1}$ and a subset $\mathcal{Q}'$ of $\mathcal{Q}$ of size k^2 so that every path in $\mathcal{P}''$ intersects every path in $\mathcal{Q}'$.

The latter outcome is the only one left to consider. Let $\mathcal{P}'' = \{P'_1, P'_2, \ldots, P'_{2k^{2k+1}}\}$ be labeled in increasing order of how they appear in $\mathcal{P}'$. In particular, the paths $\{P'_2, P'_4, \ldots, P'_{2k^{2k+1}}\}$ are all separated from each other by cuts of size q, and are all disjoint from these cuts. As we proceed from X to Y, every path in $\mathcal{Q}'$ intersects each of these separating cuts in a single vertex, and therefore intersects the (A, B)-paths in precisely the order $P'_2, P'_4, \ldots, P'_{2k^{2k+1}}$.

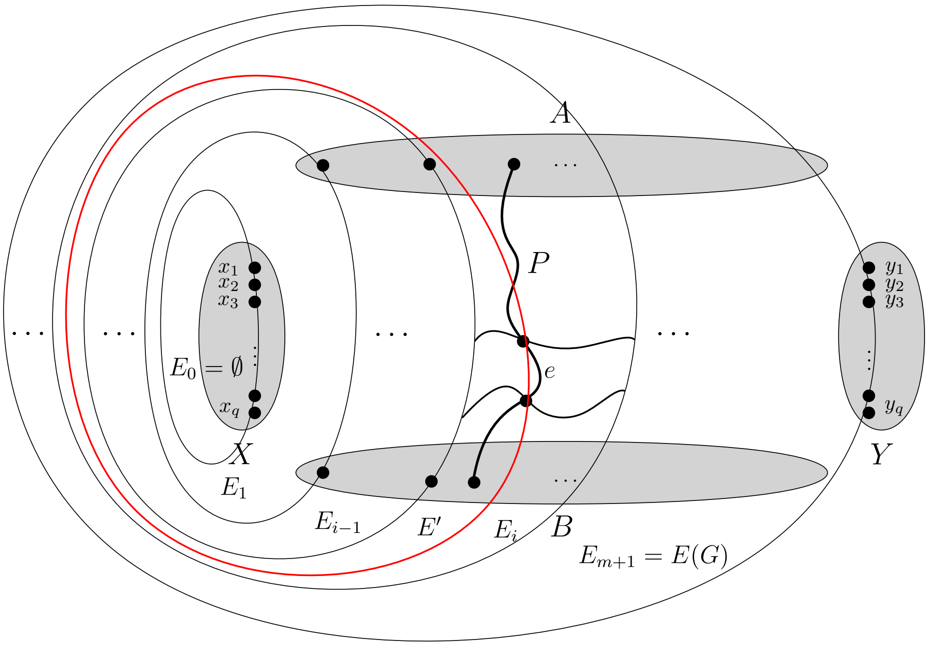

Figure 3.6.2: Illustration of the induction step of the proof of Claim 3.6.14

To complete the proof, we show that the subgraph $\{P'_2, P'_4, \ldots, P'_{2k^{2k+1}}\} \cup \mathcal{Q}'$ of G -and hence G- contains an $(k \times k)$-grid minor.

Lemma 3.6.15. Let $l \geq 2$ and $p \geq 1$ be integers. Let T be a tree with l leaves and a longest path of length at most p. Then $|V(T)| \leq (l-2)(p-1) + p + 1$.

Proof. We proceed by induction on l, the base case $l = 2$ being trivial. If $l > 2$, then T has a path P that has as one end a leaf, as the other end a vertex v of degree at least 3, and all internal vertices having degree 2. Clearly $P' := P \setminus v$ has length at most $p-2$, $T' := T \setminus V(P')$ is a tree with $l-1$ leaves, and $|V(T)| = |V(T')| + |V(P')|$. □

Lemma 3.6.16. Let $k \geq 2$, and let $(G, A, B, X, Y, \mathcal{P})$ be a $(k^{2k+1}, k^2 - 3k + 5)$-path-system. Suppose $\mathcal{Q}$ is a set of $k^2 - 3k + 5$ disjoint (X, Y)-paths such that the paths in $\mathcal{P}$ are all edge-disjoint from the paths in $\mathcal{Q}$ and every path in $\mathcal{Q}$ meets the paths in $\mathcal{P}$ in the same order. Then G contains an $(k \times k)$-grid minor.

Proof. Let $m := k^{2k+1}$, and let $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ be such that every path in $\mathcal{Q}$ meets the paths in $\mathcal{P}$ in the order $P_1, P_2, \ldots, P_m$. For each $i \in \{1, 2, \ldots, m\}$, let K_i be the graph whose vertices are the paths in $\mathcal{Q}$ an the vertices Q and Q' in $\mathcal{Q}$ are adjacent in K_i if and only if there is a subpath of P_i that has one end in Q and one end in Q' and is otherwise disjoint from all paths in $\mathcal{Q}$. Since P_i intersects all the paths in $\mathcal{Q}$, the graph K_i is connected.

Let T_i be a spanning tree of K_i. Since T_i has $(k-2)(k-1-1) + (k-1) + 2$ vertices, Lemma 3.6.15 implies that either T_i has at least k leaves or T_i has a path of length at least $k-1$. Let $(Q_{i,1}, Q_{i,2}, \ldots, Q_{i,k})$ be a sequence of vertices of T_i which are either all leaves of T_i or (in order) the vertex sequence of a path in T_i. Thus, for each $j = 2, 3, \ldots, k$, there is a subpath $P_{i,j}$ of

P_i whose ends are in the paths $Q_{i,j-1}$ and $Q_{i,j}$ and is otherwise disjoint from $Q_{i,1} \cup Q_{i,2} \cup \ldots \cup Q_{i,k}$. There are only $s = k^2 - 3k + 5$ paths in $\mathcal{Q}$ and consequently there are only $s!/(s-k)! \leq s^k \leq k^{2k}$ different possible sequences $(Q_{i,1}, Q_{i,2}, \ldots, Q_{i,k})$. Thus, some sequence occurs at least $m/s^k \geq k$ times. Let $(Q_1, Q_2, \ldots, Q_k)$ be such a sequence, and let $i_1 < i_2 < \ldots < i_k$ be among those indices i such that:

$$(Q_{i,1}, Q_{i,2}, \ldots, Q_{i,k}) = (Q_1, Q_2, \ldots, Q_k).$$

The paths Q_m, for $m = 1, 2, \ldots k$, will be the "horizontal" paths in the $(k \times k)$-grid, while the paths P_{i_j} for $j = 1, 2, \ldots, k$ will provide the "vertical" paths in the $(k \times k)$-grid. Specifically, the vertical paths will be the subpaths of P_{i_j} from Q_{m-1} to Q_m, for $m = 2, 3, \ldots, k-1$, that are otherwise disjoint from $Q_1 \cup Q_2 \cup \ldots \cup Q_k$.

In order to see exactly the $(k \times k)$-grid minor, we contract the portions of Q_m, for $m = 2, 3, \ldots, k-1$, between the ends in Q_m of the segments of P_{i_j} joining Q_m to Q_{m-1} and Q_{m+1}. It is here that we use the fact that all Q_m meet the P_{i_j} in the same order. □

The proof of Lemma 3.6.13 is completed by observing that

$$|\{P'_2, P'_4, \ldots, P'_{2k^{2k+1}}\}| = k^{2k+1} \text{ and } |\mathcal{Q}'| = k^2 \geq k^2 - 3k + 5.$$

□

Corollary 3.6.17. Let G a graph, k, n and $m_i, i \in \{1, 2 \ldots, r\}$, be positive integers, and let G be a graph. Let $X, Y, A_i, B_i, i \in \{1, 2 \ldots, r\}$ be pairwise disjoint subsets of $V(G)$. For each $i \in \{1, 2 \ldots, r\}$ let $\mathcal{P}_i$ be a set of disjoint (A_i, B_i)-paths, so that $\bigcup_{i=1}^{r} \mathcal{P}_i$ is a set of disjoint paths. If, for $i \in \{1, 2 \ldots, r\}$, the numbers $|\mathcal{P}_i|$ and $\gamma(X, Y)$ are sufficiently large (as functions of k, n and the m_i), then either G contains an $(k \times k)$-grid minor or there exist, for $i \in \{1, 2 \ldots, r\}$ sets $\mathcal{P}'_i \subseteq \mathcal{P}_i$ and a set $\mathcal{Q}$ of (X, Y)-paths, such that (for $i \in \{1, 2 \ldots, r\}$), $|\mathcal{P}'_i| \geq m_i$, $|\mathcal{Q}| \geq n$, and $\left(\bigcup_{i=1}^{r} \mathcal{P}'_i\right) \cup \mathcal{Q}$ is a set of disjoint paths.

Proof. The proof is by induction on r.

Induction Basis: For $r = 1$ the statement of Corollary 3.6.17 is the same with the statement of Lemma 3.6.13.

Induction Hypothesis: Let $l > 1$ be a positive integer, we assume that Corollary 3.6.17 holds for $r = l - 1$.

Induction Step: We prove that Corollary 3.6.17 holds for $r = l$.
Let G a graph, k, n and $m_i, i \in \{1, 2 \ldots, l\}$, be positive integers, and let G be a graph. Let $X, Y, A_i, B_i, i \in \{1, 2 \ldots, l\}$ be pairwise disjoint subsets of $V(G)$. For each $i \in \{1, 2 \ldots, l\}$ let $\mathcal{P}_i$ be a set of disjoint (A_i, B_i)-paths, so that $\bigcup_{i=1}^{l} \mathcal{P}_i$ is a set of disjoint paths.

The induction hypothesis implies that there exist positive integers $p_1, \ldots, p_{l-1}, q$ (depended on k, n and the $m_i, i \in \{1, \ldots, l-1\}$) such that if $|\mathcal{P}_i| \geq p_i, i \in \{1, \ldots, l-1\}$ and $\gamma(X, Y) \geq q$, then either G contains an $(k \times k)$-grid minor, or there exist, for $i \in \{1, 2 \ldots, l-1\}$ sets $\mathcal{P}'_i \subseteq \mathcal{P}_i$ and a set $\mathcal{Q}$ of (X, Y)-paths, such that (for $i \in \{1, 2 \ldots, l-1\}$), $|\mathcal{P}'_i| \geq m_i$, $|\mathcal{Q}| = n$, and $\left(\bigcup_{i=1}^{l-1} \mathcal{P}'_i\right) \cup \mathcal{Q}$ is a set of disjoint paths.

From Lemma 3.6.13 for the positive integers k, q and m_l, the disjoint vertex sets X, Y, A_l, B_l and the collection of disjoint paths $\mathcal{P}_l$ there exist positive integers p_l and q' such that, if $|\mathcal{P}_l| \geq p_l$ and $\gamma(X, Y) \geq q'$ either G has an $(k \times k)$-grid minor (in which case we are done), or there is a subset $\mathcal{P}_l'$ of $\mathcal{P}_l$ of size m_l and a set $\mathcal{Q}'$ of (X,Y)-paths of size q such that $\mathcal{P}_l' \cup \mathcal{Q}'$ is a set of disjoint paths.

Let us now consider the latter outcome. We first delete every (X,Y)-path in G that is not in $\mathcal{Q}'$ and apply the induction hypothesis for the positive integers k, n and $m_i, i \in \{1,\ldots,l-1\}$, the disjoint vertex sets $X, Y, A_i, B_i, i \in \{1,\ldots,l-1\}$, and the sets of disjoint paths $\mathcal{P}_i$ $i \in \{1,2\ldots,l\}$. Either G has a $(k\times k)$-grid minor (in which case we are done), or there exist, for $i \in \{1,2\ldots,l-1\}$ sets $\mathcal{P}_i' \subseteq \mathcal{P}_i$ and a set $\mathcal{Q}$ of (X,Y)-paths, such that (for $i \in \{1,2\ldots,l-1\}$), $|\mathcal{P}_i'| \geq m_i$, $|\mathcal{Q}| = n$, and $\left(\bigcup_{i=1}^{l-1}\mathcal{P}_i'\right) \cup \mathcal{Q}$ is a set of disjoint paths, since $\mathcal{Q} \subseteq \mathcal{Q}'$ we are done.

Induction Conclusion: Corollary 3.6.17 holds. □

Corollary 3.6.18. Let G be a graph, $k, m_i, i \in \{1,2\ldots,r\}$ and $n_j, j \in \{1,2,\ldots,s\}$ be positive integers, and let G be a graph. Let $(A_i,B_i), i \in \{1,2\ldots,r\}$ and $(X_j,Y_j), j \in \{1,2,\ldots,s\}$ be pairwise disjoint subsets of $V(G)$. For $i \in \{1,2\ldots,r\}$, let $\mathcal{P}_i$ be a set of (A_i,B_i)-paths so that $\bigcup_{i=1}^{r}\mathcal{P}_i$ is a set of disjoint paths. If the numbers $|\mathcal{P}_i|$ for $i \in \{1,2\ldots,r\}$, and $\gamma(X_j,Y_j)$ for $j \in \{1,2,\ldots,s\}$ are sufficiently large (as functions of k, m_i and n_j), then either G contains a $(k\times k)$-grid minor or there exist $\mathcal{P}_i' \subseteq \mathcal{P}_i$ for $i \in \{1,2\ldots,r\}$ and for each $j \in \{1,2,\ldots,s\}$ a set $\mathcal{Q}_j$ of (X_j,Y_j)-paths, such that $(\forall i \in \{1,2,\ldots,r\})[|\mathcal{P}_i'| \geq m_i]$, $(\forall j \in \{1,2,\ldots s\})[|\mathcal{Q}_j| \geq n_j]$ and $\left(\bigcup_{i=1}^{r}\mathcal{P}_i'\right) \cup \left(\bigcup_{j=1}^{s}\mathcal{Q}_j\right)$ is a set of disjoint paths.

Proof. The existence is established by induction on s.

Induction Basis: For $s = 1$ the statement of Corollary 3.6.18 is the same with the statement of Corollary 3.6.17.

Induction Hypothesis: Let $l > 1$ be a positive integer, we assume that Corollary 3.6.18 holds for $s = l-1$.

Induction Step: We prove that Corollary 3.6.18 holds for $s = l$.
From the induction hypothesis there exist positive integers $p_1,\ldots,p_r,q_1,\ldots,q_{l-1}$ (depended on k, m_i and n_j) such that if $|\mathcal{P}_i| \geq p_i$ for $i \in \{1,2\ldots,r\}$, and $\gamma(X_j,Y_j) \geq q_j$ for $j \in \{1,2,\ldots,l-1\}$, either G contains a $(k\times k)$-grid minor or there exist $\mathcal{P}_i' \subseteq \mathcal{P}_i$ for $i \in \{1,2\ldots,r\}$ and for each $j \in \{1,2,\ldots,l-1\}$ a set $\mathcal{Q}_j$ of (X_j,Y_j)-paths, such that $(\forall i \in \{1,2,\ldots,r\})[|\mathcal{P}_i'| \geq m_i]$, $(\forall j \in \{1,2,\ldots l-1\})[|\mathcal{Q}_j| \geq n_j]$ and $\left(\bigcup_{i=1}^{r}\mathcal{P}_i'\right) \cup \left(\bigcup_{j=1}^{l-1}\mathcal{Q}_j\right)$ is a set of disjoint paths.

From the Corollary 3.6.17, for the positive integers k, n_l and $p_i, i \in \{1,2\ldots,r\}$, the pairwise disjoint vertex sets $X_l, Y_l, A_i, B_i, i \in \{1,2\ldots,r\}$ and the sets of disjoint (A_i,B_i)-paths $\mathcal{P}_i$ for $i \in \{1,2\ldots,r\}$ there exist positive integers $p_1',\ldots,p_r',q'$, such that if $(\forall i \in \{1,\ldots,r\})|\mathcal{P}_i| \geq p_i'$ and $\gamma(X_l,Y_l) \geq q'$, then either G contains an $(k\times k)$-grid minor (in which case we are done) or there exist, for $i \in \{1,2\ldots,r\}$ sets $\mathcal{P}_i' \subseteq \mathcal{P}_i$ and a set $\mathcal{Q}_l$ of (X_l,Y_l)-paths, such that (for $i \in \{1,2\ldots,r\}$), $|\mathcal{P}_i'| = p_i$, $|\mathcal{Q}_l| = n_l$, and $\left(\bigcup_{i=1}^{r}\mathcal{P}_i'\right) \cup \mathcal{Q}_l$ is a set of disjoint paths.

Let us now consider the latter outcome. We apply the induction hypothesis for the positive integers $k, n_j, j \in \{1,\ldots l-1\}$ and $m_i, i \in \{1,\ldots,r\}$, the disjoint vertex sets $X_j, Y_j, j \in \{1,\ldots l-1\}$, the disjoint vertex sets $A_i, B_i, i \in \{1,\ldots,l-1\}$, and the sets of disjoint paths $\mathcal{P}_i'$ $i \in \{1,\ldots,r\}$ (which are sufficient large). There exist positive integers $q_1,\ldots,q_{l-1}$ (depended on k, m_i and n_j) such that if $\gamma(X_j,Y_j) \geq q_j$ for $j \in \{1,2,\ldots,l-1\}$, then either G has a $(k\times k)$-grid minor (in which case we are done), or there exist $\mathcal{P}_i'' \subseteq \mathcal{P}_i$ for $i \in \{1,2\ldots,r\}$ and for each $j \in \{1,2,\ldots,l-1\}$ a set $\mathcal{Q}_j$ of (X_j,Y_j)-paths, such that $(\forall i \in \{1,2,\ldots,r\})[|\mathcal{P}_i''| \geq m_i]$, $(\forall j \in \{1,2,\ldots l-1\})[|\mathcal{Q}_j| \geq n_j]$ and $\left(\bigcup_{i=1}^{r}\mathcal{P}_i'\right) \cup \left(\bigcup_{j=1}^{l-1}\mathcal{Q}_j\right)$ is a set of disjoint paths. Since

$(\forall i \in \{1,\dots,r\})[\mathcal{P}''_i \subseteq \mathcal{P}'_i]$, we have that $\left(\bigcup_{i=1}^{r} \mathcal{P}''_i\right) \cup \left(\bigcup_{j=1}^{l} \mathcal{Q}_j\right)$ is a set of disjoint paths, and thus we are done.

Induction Conclusion: Corollary 3.6.18 holds. □

Lemma 3.6.19. Let $h \geq k \geq 2$ be positive integers and let G be a graph. Then either G has branch-width at most h or there is a subset E of $E(G)$ such that:

(i) $\gamma(E) = h$;

(ii) $\Gamma(E)$ is k-connected in $G \setminus E$;

(iii) there is a path P such that $E(P) \subseteq E$ and P contracts in G onto $\Gamma(E)$.

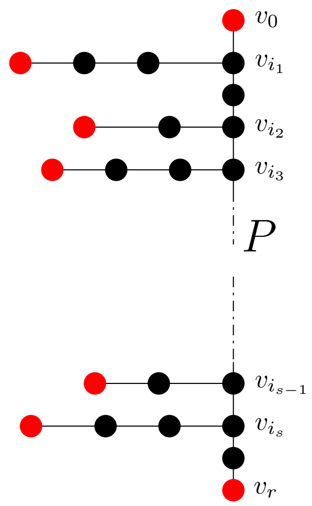

Figure 3.6.3: The correct image for the path P on Lemma 3.6.19 is the following: $P = (v_0, v_1, \dots, v_r)$ has its ends $v_0, v_r \in \Gamma(E)$ and there are indices $0 < i_1 < \dots < i_s < r$ and a set $\{P_1, P_2, \dots, P_s\}$ of $s = \gamma(E) - 2$ disjoint paths, such that for each $j \in \{1,\dots,s\}$ the path P_j joins v_{i_j} to a vertex of $\Gamma(E) \setminus \{v_0, v_r\}$, and meets P only in v_{i_j}.

Proof. We suppose that the branch-width of the graph G is greater than h. Let E^* be a maximal subset of $E(G)$ such that:

(i) $\gamma(E^*) \leq h$;

(ii) if (T,τ) is a branch-decomposition such that the set E^* is displayed by an edge of T, say e_1, then some edge e'_1 in the component of $T \setminus e_1$ not containing $\tau(E^*)$ has width greater than h, i.e., $\gamma(e'_1) > h$;

(iii) there is a path P in G such that $E(P) \subseteq E^*$ and P contracts in G onto $\Gamma(E^*)$.

To see that such a choice of a set is possible, it suffices to observe that $\emptyset$ satisfies the above conditions. Moreover, condition (ii) implies that $E^* \neq E(G)$.

Claim 3.6.20. $\gamma(E^*) = h$

Proof of Claim 3.6.20. Let us suppose toward a contradiction that $\gamma(E^*) < h$. Let $v \in \Gamma(E^*)$ be an end of P. By the definition of the set $\Gamma(E^*)$, it follows that there exists an edge $e \in E(G) \setminus E^*$ incident with v. The submodularity of γ implies that:

$$\gamma(E^* \cup \{e\}) \leq 1 + \gamma(E^*) \leq h$$

, and there is a path P', whose edges are all in $E^* \cup \{e\}$, that contracts onto $\Gamma(E^* \cup \{e\})$. Since $E^* \subsetneq E^* \cup e$, it follows by the maximality of E^* that there exist a branch-decomposition of G, say (T, τ), such that the set $E^* \cup \{e\}$ is displayed by an edge, say e_1, and for every edge e_2 in the component of $T \setminus e_1$ not containing $\gamma(e)$ the width of e_2 is $\leq h$ i.e., $\gamma(e_2) \leq h$.

Let T_1, T_2 be the components of $T \setminus e_1$, so that the function τ maps the edge e at a leaf of the component T_1. We now proceed to the construction of a new branch-decomposition (T', τ') of the graph G which will give us the desired contradiction. The branch-decomposition (T', τ') is obtained as follows: The vertices of T' are those of T, with two new vertices t_1, t_2 that are adjacent in T'. In addition, t_2 is adjacent to the two vertices of T incident with the edge e_1. All other adjacencies in T' are in complete agreement with adjacency in $T \setminus e_1$.

We set $\tau'(e) = t_1$, and for all the other edges e' of G we set $\tau'(e') = \tau(e')$. Clearly the width of the edge $\{t_1, t_2\}$ of the tree T is at most 2, the edge joining T_1 to t_2 has width $\gamma(E^*)$ which is less than h by our assumption, the edge joining t_2 to T_2 has width $\gamma(E^* \cup \{e\}) \leq h$, and every other edge e' in T_2 we have $\tau'(e') = \tau(e')$ that is, at most h contradicting to (ii). Thus, the claim holds. ☐

Claim 3.6.21. *The set of vertices $\Gamma(E^*)$ is k-connected in $G \setminus E^*$.*

Proof of Claim 3.6.21. Let us suppose towards a contradiction that the claim does not hold. Let A, B be subsets of $\Gamma(E^*)$ with $|A| = |B| = s$, such that s is as small as possible and they witness our assumption, that is, there is no set of s disjoint (A, B)-paths in the graph $G \setminus E^*$ with each path disjoint from the $\Gamma(E^*)$ except its ends.

Let H be the subgraph of G which is induced by the edges of $E(G) \setminus E^*$, with the vertices of $\Gamma(E^*) \setminus (A \cup B)$ deleted. Since we need to work in both G and H for the next few paragraphs, we include as a superscript, the name of the graph (G or H) in which we are determining $\Gamma(F)$, for some set F of edges.

Let $E' \subseteq E(H)$ be such that $S = \Gamma^H_{(A,B)(E')}$ is a smallest set that separates A from B in H. Clearly, $|S| = s - 1$ (by the choice of s). By Menger's Theorem 1.2.73, there is a set $\mathcal{Q}$ of $s - 1$ disjoint (A, S)-paths in H.

Claim 3.6.22. $\Gamma^G(E^* \cup E') \subseteq (\Gamma^G(E^*) \setminus A) \cup S$.

Proof. Let z be an arbitrary but fixed vertex in $\Gamma^G(E^* \cup E')$, then z is incident with an edge $\bar{e} \notin E^* \cup E'$ and an edge $e \in E^* \cup E'$. If $e \in E^*$, then $z \in \Gamma^G(E^*)$, while if $e \in E'$, then $z \in \Gamma^H(E') \subseteq S$. As z was arbitrary it follows that $\Gamma^G(E^* \cup E') \subseteq \Gamma^G(E^*) \cup S$ ☐

Let $u \in A \setminus S$. Then u is not incident with an edge of $E(H) \setminus E'$, so every edge in G incident with u is in $E^* \cup E'$. Hence $u \notin \Gamma^G(E^* \cup E')$.

It follows immediately from the Claim 3.6.22 that $\gamma_G(E^* \cup E') < h$ (since $A \subseteq \Gamma^G(E^*)$ and $|S| < |A|$). The set $\mathcal{Q}$ of the disjoint (A, S)-paths shows that there is a path P' using only edges in $E^* \cup E'$ that contracts to $\Gamma(E^* \cup E')$. Since $E^* \subsetneq E^* \cup E'$, the maximality of E^* implies the existence of a branch-decomposition, say (T_A, f_A), of G with the property that, for some edge τ_A of T_A, $f_A(E^* \cup E')$ is contained in the set of leaves of the one component, say T_A^1, of $T_A \setminus \tau_A$ and $f_A(E(G) \setminus (E^* \cup E'))$ is contained in the set of leaves of the other component, say T_A^2, of $T_A \setminus \tau_A$ so that, $(\forall \tau \in E(T_A^2))[\gamma_G(\tau) \leq h]$.

In exactly the same way, we can proceed with B in place of A. For ease of notation, let $E'' := E(H) \setminus E'$. Then there are a branch-decomposition (T_B, f_B) of G and an edge τ_B of T_B such that $f_B(E^* \cup E'')$ is contained in the set of leaves of the one, say T_B^1, of $T_B \setminus \tau_B$ and $f_B(E(G) \setminus (E^* \cup E''))$ is contained in the set of leaves of the other component, say T_B^2, of $T_B \setminus \tau_B$ so that, $(\forall \tau \in E(T_B^2))[\gamma_G(\tau) \leq h]$.

We now proceed to the description of a third branch-decomposition which will give us the desired contradiction. Let T be any cubic tree containing a vertex t such that the graph $T \setminus t$ has three components, one isomorphic to T_A^2, one isomorphic to T_B^2, and the third, say T_3, with at least $|E^*|$ leaves that are also leaves of T. Moreover, the edge joining T_A^2 to t is incident with the end of τ_A in T_A^2, and similarly the edge joining T_B^2 to t is incident with the end of τ_B in T_B^2. We define the function f as follows: $(\forall a \in E'')[f(a) := f_A(a)]$, $(\forall b \in E')[f(b) := f_B(b)]$ and for each $e \in E^*$, $f(e)$ is any leaf of the component T_3 that is also a leaf of T, chosen so that f is an injection. It is easy to see that if τ is the edge of T joining T_3 to t, then E^* is displayed by τ, while if τ' is any edge of $T_A^2 \cup T_B^2$, then $\gamma(\tau') \leq h$, thereby contradicting the choice of E^*. Thus, indeed the set $\Gamma(E^*)$ is k-connected in $G \setminus E^*$ and the proof of Claim 3.6.21 is complete. ☐

The proof of Lemma 3.6.19 is completed since we have found the desired set which is the set E^* as it is witnessed by condition (iii) for the set E^* and Claims 3.6.20, 3.6.21 ☐

We are now ready to prove the Excluded Grid Theorem.

Proof of Theorem 3.6.4. Let k be an arbitrary but fixed positive integer, by Corollary 3.6.18 (with $r = 0, s = 2k^2$ and each $n_i = 1$), there is an integer $F(k)$ such that:

If G is a graph and $\{X_j, Y_j | 1 \leq j \leq 2k^2\}$ are disjoint subsets of $V(G)$ with the property that $(\forall j \in \{1, 2, \ldots, 2k^2\})[\gamma(X_j, Y_j) \geq F(k)]$, then

- either the graph G has an $(k \times k)$-grid minor
- or there is for each $j \in \{1, 2, \ldots, 2k^2\}$ a (X_j, Y_j)-path P_j such that the set $\{P_1, P_2, \ldots, P_{2k^2}\}$ is a collection of disjoint paths.

Let G be an arbitrary but fixed graph such that $\mathbf{bw}(G) > 2k^2 F(k)$. By Lemma 3.6.19, there is a subset E^* of $E(G)$ such that:

(i) $\gamma(E^*) = 2k^2 F(k)$;

(ii) the set of vertices $\Gamma(E^*)$ is $\big(2F(k)\big)$-connected in $E(G) \setminus E^*$ and

(iii) there is a path P such that $E(P) \subseteq E^*$ and P contracts in G onto the set $\Gamma(E^*)$.

Let $\{v_1, v_2, \ldots, v_{2k^2 F(k)}\}$ be the vertices of the set $\Gamma(E^*)$ enumerated in the order in which they appear in the contraction of P and consider the partition $V_1, V_2, \ldots, V_{k^2}$ of $\Gamma(E^*)$ obtained by putting $2F(k)$ consecutive vertices into each of these sets, i.e. for $l \in \{1, \ldots, k^2\}$ $V_l = \{v_m | l \leq m \leq m + 2k - 1\}$.

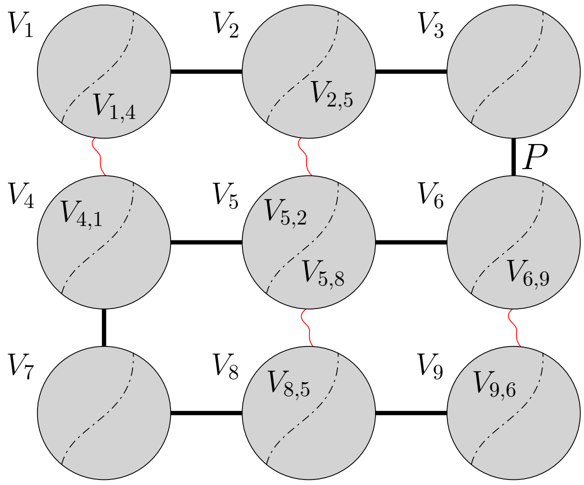

Figure 3.6.4: In the proof of the excluded grid Theorem 3.6.4 we are trying to deduce the desired $(k \times k)$-grid minor as follows: We are using the path P to take a Hamiltonian path of the $(k \times k)$-grid, and each of the k^2 disjoint vertex sets $V_i, i \in \{1, \dots, k^2\}$ corresponds to one vertex of the grid. Having the Hamiltonian path, to complete the grid minor we need to connect some vertex sets\vertices of the grid with disjoint paths\edges of the grid, if we cannot find the desired disjoints paths to do that and show fail to constuct the grid minor by the above way then Corollary 3.6.18 ensure the existence of an $(k \times k)$-grid minor on G. The case illustrated on the above figure is for $k = 3$, the desired paths to complete the grid minor have been drawn by red color.

Thinking of the path P as giving us a Hamiltonian path in the $(k \times k)$-grid minor that starts and ends in different corners of the grid, we see that we only need to connect some of the sets V_i in pairs in order to make the $(k \times k)$-grid minor. We will attempt to join these pairs of sets using only edges in $E(G)\ E^*$, in order not to interfere with the Hamiltonian path. For each $i \in \{1, \dots, k^2\}$ the set V_i is to be joined to at most two others sets.

For each $i \in \{1, \dots, k^2\}$ let j and j' be the indices for which we are trying to join V_i to both V_j and $V_{j'}$, note that for some indices i there exist only one such index j. Arbitrarily partition the set V_i into two sets $V_{i,j}$ and $V_{i,j'}$.

Consider the pairs $(V_{i,j}, V_{j,i})$. Since $\Gamma(E^*)$ is $\big(2F(k)\big)$-connected in $E \backslash E^*$, $V_{i,j}, V_{j,i} \subseteq \Gamma(E^*)$ and $|V_{i,j}| = |V_{j,i}| = F(k)$ it follows that $\gamma(V_{i,j}, V_{j,i}) \geq F(k)$. Hence, by the choice of the integer $F(k)$, it follows that either G has an $(k \times k)$-grid minor or we can find disjoint paths in $E(G) \setminus E^*$ between the required pairs so that we can complete the $(k \times k)$-grid minor in G. In either case, G has an $(k \times k)$-grid minor as required. $\square$

The initial upper bound [108] on the tree-width sufficient for the $(k \times k)$-grid minor was enormous; *«it involved iterated exponentiation where the number of iterations also involved iterated exponentiation (and so on, twice more)»* as remarked by Robertson, Seymour and Thomas in [115] where they improved that ([115, Theorem 1.6]) to $g(k) \leq 20^{2k^5}$. In the same paper they note that, by use of a probabilistic argument, they have proved the existence of graphs which have no $(k \times k)$-grid minor and have tree-width at least proportional to $k^2 \log k$ and they also conjectured *«that $\mathcal{O}(k^2 \log k)$ is closer to the right answer»* than the bound that they obtained.

As we have already mention the first short proof of the Excluded Grid Theorem was given by Diestel, Jensen, Gorbunov, and Thomassen [34] (see also [32]), but the upper bound that they obtained there, where $2^{5k^5 \log k}$, which is slightly worse than the bound provided by Robertson, Seymour, and Thomas [115]. Kawarabayashi and Kobayashi [69] proved that $g(k) \in 2^{\mathcal{O}(k^2 \log k)}$, and Leaf and Seymour [86] proved that $g(k) \in 2^{\mathcal{O}(k \log k)}$. The function $g(k)$ was first shown to be polynomial by Chekuri and Chuzhoy [19] who showed $g(k) \in \mathcal{O}(k^{98} poly \log k)$, this was further improved to $g(k) \in \mathcal{O}(k^{36} poly \log k)$ by Chuzhoy [23] and later -again by Chuzhoy [24]- to $g(k) \in \mathcal{O}(k^{19} poly \log k)$ which is -as far as we know- is the best known upper bound. Demaine, Hajiaghayi, and Kawarabayashi [28] conjectured that all graphs with tree-width $\Omega(k^3)$ have a $\big(\Omega(k) \times \Omega(k)\big)$-grid minor and that this bound is tight.

The Excluded Grid Theorem has an interesting application. Recall the Erdős-Pósa theorem (Theorem 4.1.5), it is natural to ask if a similar result holds for other structures instead of cycles.

Definition 3.6.23 (Erdős-Pósa property)**.** A class $\mathcal{H}$ of graphs is said to have the *Erdős-Pósa property* if and only if the number of vertices in a graph which is needed to cover all its subgraphs in $\mathcal{H}$ is bounded by a function of its maximum number of disjoint subgraphs in $\mathcal{H}$.

Robertson and Seymour [108] proved using the Excluded Grid Theorem the following generalization of the Erdős-Pósa theorem.

Theorem 3.6.24 (Robertson and Seymour [108])**.** Let H be a fixed connected graph, and consider the class $\mathcal{H}$ of all graphs which contract to a graph isomorphic to H. $\mathcal{H}$ has the Erdős-Pósa property if and only if H is planar.

The Erdős-Pósa property is well-studied and has several applications. For a survey of results we refer the interested reader in [101].

3.6.2 Well-quasi-ordering graphs which exclude a fixed planar graph as a minor

Proof of Theorem 3.6.2. Let H be an arbitrary but fixed planar graph and let us denote by $\mathcal{G}$ the set of all graphs with no minor isomorphic with H.

Since from Theorem 3.5.9 there exists an integer k such that the graph H is isomorphic to a minor of the $(k \times k)$-grid, it follows from the transitivity of the minor relation that no graph in $\mathcal{G}$ has a minor isomorphic to the $(k \times k)$-grid.

By the Theorem 3.6.4, there exist an integer $f(k)$ such that every graph with branch-width greater or equal to $f(k)$ has a minor isomorphic to the $(k \times k)$-grid, thus $(\forall G \in \mathcal{G})[\mathbf{bw}(G) < f(k)]$.

Hence by Theorem 3.4.2 it follows that the set $\mathcal{G}$ is well-quasi-ordered by the minor relation. Since H was an arbitrary planar graph the proof is complete. □

Proof of Theorem 3.6.1. Let $G_1, G_2, \ldots$ be an arbitrary but fixed infinite sequence of graphs, such that the graph G_1 is planar. If there exist a positive integer $j > 1$ such that G_1 is isomorphic to a minor of G_j, we are done. If for every positive integer $j > 2$ the graph G_j has no minor isomorphic to the planar graph G_1 then by Theorem 3.6.2 the set of graphs $\{G_j | j > 1\}$ is well-quasi-ordered by the minor relation (as a subset of the well-quasi-ordered set of all graphs with no minor isomorphic to the planar graph G_1). Thus the sequence $(G_j)_{j \geq 2}$ is an infinite sequence of a well-quasi-ordered set, and hence must contain at least one good pair of graphs, that is, there exist i, j with $j > i$ such that that G_i is isomorphic to a minor of G_j. Since $G_1, G_2, \ldots$ was an arbitrary infinite sequence of graphs, our proof is complete. □

The path-width of a graph is the minimum value k such that the graph can be obtained from a sequence of graphs $G_1, \dots, G_r$ each of which with at most $k+1$ vertices, by identifying some vertices of G_i pairwise with some vertices of G_{i+1} $(1 \leq i < r)$. As a forerunner of the Excluded Grid Theorem and Theorem 3.6.2, Robertson and Seymour [105] proved in 1983 the following:

Theorem 3.6.25 (Robertson and Seymour [105])**.** For every forest F, there exist a positive integer $n(|V(F)|)$ such that every graph with path-width at least $n(|V(F)|)$ has a minor isomorphic to F.

In 1991 Bienstock, Robertson, Seymour, and Thomas [11] obtained a shorter proof and they brown down the bound of path-width to the best possible. A shorter proof which re-obtain the optimal bound was given by Diestel [31] in 1995.

Theorem 3.6.26 (Bienstock, Robertson, Seymour, and Thomas [11], Diestel [31])**.** For every forest F, every graph with path-width at least $|V(F)| - 1$ has a minor isomorphic to F.

Thomas [120] proved the following generalization of Theorem 3.6.1.

Theorem 3.6.27 (Thomas [120])**.** If $G_1, G_2, \dots$ is any infinite sequence of graphs, such that G_1 is a finite planar graph and for each $i > 1$ the graph G_i is finite or infinite, then there exist i, j with $j > i \geq 1$ such that G_i is isomorphic to a minor of G_j.

3.7 A Kuratowski theorem for general surfaces

We present a proof of Theorem 3.1.5 which states that embeddability in any fixed surface can be characterized by forbidding finitely many graphs as minors. As we mentioned in the introduction of this chapter, Theorem 3.1.5 was conjectured by Erdős and König in the 1930s. The first positive result was the finiteness of the set of forbidden minors for the projective plane which was proved by Glover and Huneke [58] in 1978. One year later Glover, Huneke, and San Wang [59] presented a list of 103 minimal forbidden subgraphs for the projective plane. Finally, Archdeacon [2, 4] proved in 1980 that this list is complete. The set of forbidden minors for the projective plane contains 35 graphs (see [92, p. 198]). Archdeacon and Huneke [3] gave a constructive proof of the finiteness of the set of forbidden minors for any non-orientable surface. For orientable surfaces, Bodendiek and Wagner [16] proved the finiteness of the set of forbidden minors for the torus and finally a non-constructive proof for general surfaces was given by Roberson and Seymour in [110]. For the case of general surfases see also [17]. The first short proof of Theorem 3.1.5 was given by Thomassen [122]. Until today, the only other than the sphere surface for which the forbidden minors are known is the projective plane, the number of forbidden minors appears to grow enormously even for simple surfaces, for example it was recently announced in [94] that the forbidden minors of torus are at least 17.523.

The proof presented in this section is due to Geelen, Richter, and Salazar [54], and the presentation follows [33]. Our main ingredient considers the embedding of grids in surfaces. We prove that when a large grid is embedded in a surface, most of the grid is embedded in a planar way. This has as an immediate consequence that for any fixed surface Σ the minor-minimal graphs which are not embeddable in Σ cannot contain arbitrary large grid minors, then the Excluded Grid Theorem implies that their branch-width is bounded and finally the desired result comes combining the well-quasi-ordering of graphs of bounded branch-width (Theorem 3.4.2) by the minor relation and the fact that the set of forbidden minors for any fixed surface is an antichain, and since there exist no infinite antichains in well-quasi-ordered spaces this set is finite.

3.7.1 Embedding grids in surfaces

Definition 3.7.1 (hexagonal grid, central face of hexagonal grid, canonical subgrid, ring of hexagonal grid)**.** Let k be a positive integer, the *hexagonal grid* k -denoted by H_k- is a finite subgraph of the hexagonal tiling of the Euclidean plane. We let H_1 be a cycle of length 6. For each $k \geq 2$, we define H_k as the union of H_{k-1} and all those 6-cycles in the hexagonal tiling of the Euclidean plane which have common neighbors with H_{k-1}. The face corresponding to the central vertex of the dual of an hexagonal grid is its *central face*. Given positive integers $m < k$, the subgrid H_m of H_k will be said to be *canonical* if and only if its central face coincidences with the central face of H_k. We denote by S_m the perimeter cycle of the canonical subgrid H_m in H_k. The *ring* R_m the subgraph of H_k formed by S_m and S_{m+1} and the edges between them. Clearly, for each $k > 1$, H_k is a subdivision of a 3-connected graph.

Comment 3.7.2. *For each positive integer k, the hexagonal grid H_k contains an $(k \times k)$-grid minor and is a subgraph of the $(4k \times 2k)$-grid.*

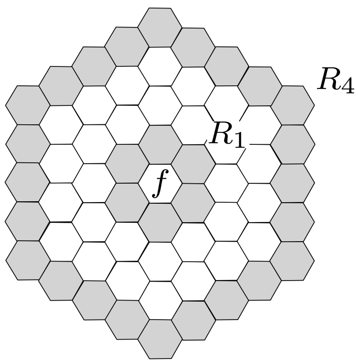

Figure 3.7.1: The hexagonal grid H_5 and its rings R_1, R_4. The central face of H_5 is denoted by f.

Lemma 3.7.3. For every surface Σ there exists an integer k such that no graph that is minor-minimal with the property of not being embeddable in Σ contains a minor isomorphic to the $(k\times k)$-grid.

Recall Proposition 1.2.59 and notice that since the hexagonal grid is a subcubic graph, whenever we exclude a hexagonal grid as a topological minor of a graph we exclude it also as a minor. Moreover, since the hexagonal grid is a planar graph, Theorem 3.5.9 implies that whenever we exclude as a minor a hexagonal grid we exclude also a square grid. Thus, in order to prove Lemma 3.7.3 we can prove the following:

Lemma 3.7.4. For every surface Σ there exist an integer r such that no graph that is minor-minimal with the property of not being embeddable in Σ contains a topological minor isomorphic to H_r.

Proof. Let Σ be an arbitrary but fixed surface, and let ε denote the Euler genus of Σ, i.e. $\varepsilon := \varepsilon(\Sigma)$ Let r be large enough to ensure that H_r contains $\varepsilon+3$ disjoint copies of H_{m+1}, where $m := 3\varepsilon+4$. Let G be a graph that cannot be embedded in Σ and is minor-minimal with that property. The rest of the proof lies in showing that $H_r \nleq_{tm} G$.

Let $e' = \{u', v'\}$ be an arbitrary but fixed edge of G, the choice of G implies that the graph $G \setminus e'$ is embeddable in Σ. Let σ' be an embedding of $G \setminus e'$ in Σ. Choose a face with u' in its boundary and a face with v' in its boundary. Cut a disc out of each face and add a handle between the two holes. We denote by Σ' the resulting surface. By Lemma 1.4.24, the Euler genus of Σ' is $\varepsilon + 2$. Embedding e' along this handle, extend σ' to an embedding of G in Σ'.

Let us suppose towards a contradiction that $H_r \leq_{tm} G$, let H be a subgraph of G and $f : H_r \to H$ be a function which witness our assumption. i.e., f maps the vertices of H_r to the corresponding branch vertices of H, and its edges to the corresponding paths in H between those vertices

Claim 3.7.5. *The hexagonal grid H_r has an hexagonal subgrid H_m (not necessarily canonical) whose hexagonal face boundaries correspond (by $\sigma' \circ f$) to cycles in Σ' that bound disjoint open discs there.*

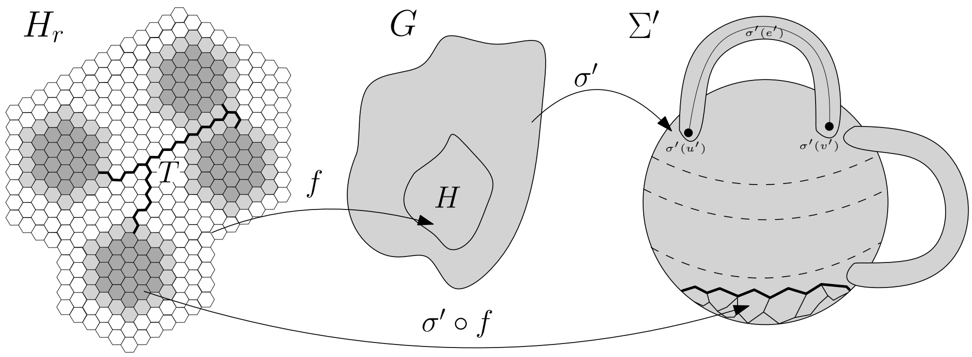

Figure 3.7.2: On the left hand side disjoint copies of H_m (for $m = 3$) are linked up by a tree T in the rest of H_r. In the middle is illustrated the graph G and its subgraph H which contains the hexagonal grid H_r as a topological minor. In the right hand side is illustrated the surface Σ' in which the graph G is embedded via the embedding σ' and the union of the faces of $f(H_m)$ for the special copy of H_m in H_r -the existence of which guaranteed Claim 3.7.5- is a disc.

Proof of Claim 3.7.5. By the choice of r, we can find $\varepsilon + 3$ disjoint copies of H_{m+1} in H_r. The canonical subgrids H_m of these H_{m+1} are not only disjoint, but sufficiently spaced out in H_r that their deletion leaves a tree $T \subseteq H_r$ which sends an edge to each of these copies of H_m in H_r.

Hence whenever we pick one hexagon from each of these H_m and delete the images $\mathcal{C}$ of those hexagons in Σ', the component D_0 of the remainder of Σ' that contains $(\sigma' \circ f)(T)$ meets all those $\mathcal{C}$ in its boundary. By Lemma 1.4.25 and $\varepsilon(\Sigma') = \varepsilon + 2$, therefore, it can not be true that none of our circles $\mathcal{C}$ bounds a disc in Σ' that is disjoint from $(\sigma' \circ f)(T)$. Hence for one of our copies of H_m in H_r, the image of every hexagon in S' bounds an open disc that is disjoint from $(\sigma' \circ f)(T)$.

Let us show that these discs are disjoint. If not, then one of them, say D, contains a point, say x, from the boundary of another such disc. But then D also contains $(\sigma' \circ f)(T)$, contrary to assumption, because we can walk from x to $(\sigma' \circ f)(T)$ in $(\sigma' \circ f)(H_r) \subseteq \Sigma'$ avoiding the boundary of D. □

From now on, we shall work with this fixed H_m and no more consider its supergraph H_r. We write $C_i := f(S_i)$ for the images in G of the concentric cycles S_i of this H_m, $(i = 1, \ldots, m)$. Let

$e = \{u, v\}$ be an arbitrary but fixed edge of C_1 and choose an embedding σ of $G \setminus e$ in Σ.

Claim 3.7.6. *One of the $\varepsilon + 1$ disjoint rings $R_{3i+2}, (i = 0, \ldots, \varepsilon)$ in our H_m, say R_k, has the property that its hexagons correspond (by $\sigma \circ f$) to circles in Σ that bound disjoint open discs there.*

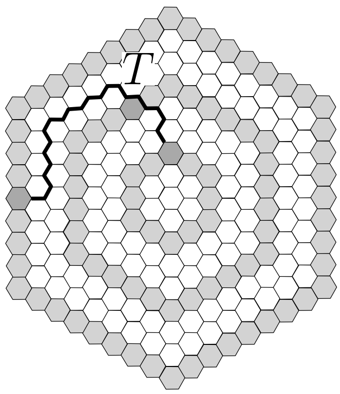

Figure 3.7.3: A tree linking up hexagons in H_m selected from the rings $R_2, R_5, R_8, \ldots$

Proof of Claim 3.7.6. For each $i \in \{0, \ldots, \varepsilon\}$ we choose one arbitrary but fixed hexagon of the ring R_{3i+2}. Let $\mathcal{C}$ be the set of the images (via $\sigma \circ f$) of those hexagons in Σ.

If we delete those hexagons in H_m, in the remainder of H_m there exist a tree T which is linking up all those hexagons. If we delete the images $\mathcal{C}$ of those hexagons in Σ, the component D_0 of $\Sigma \setminus \mathcal{C}$ that contains $(\sigma \circ f)(T)$ meets all those $\mathcal{C}$ in its boundary. By Lemma 1.4.25 and $\varepsilon(\Sigma) = \varepsilon$, therefore, it can not be true that none of our circles in $\mathcal{C}$ bounds a disc in Σ that is disjoint from $(\sigma \circ f)(T)$ because then we would have $\varepsilon \geq |\mathcal{C}| = \varepsilon + 1$. Hence for one of these $\varepsilon + 1$ rings, say R_k, the image of every hexagon of this ring in S bounds an open disc that is disjoint from $(\sigma \circ f)(T)$.

Let us show that these discs are disjoint. If not, then one of them, say D, contains a point, say x, from the boundary of another such disc. But then D also contains $(\sigma \circ f)(T)$, contrary to assumption, because we can walk from x to $(\sigma \circ f)(T)$ in $(\sigma \circ f)(H_m) \subseteq \Sigma$ avoiding the boundary of D. □

Let $R \supseteq (\sigma \circ f)(R_k)$ be the closure in Σ of the union of those discs, which is a cylinder in Σ and observe that one of the two boundary circles of R is the image under σ of the cycle $C := C_{k+1}$ in G to which f maps the perimeter cycle S_{k+1} of our special ring $R_k \subseteq H_m$.

Let $H' := f(H_{k+1}) \subseteq G$, where H_{k+1} is canonical in our H_m. Recall that $\sigma' \circ f$ maps the hexagons of H_{k+1} to circles in Σ' bounding disjoint open discs there. The closure in Σ' of the union of these discs is a disc D' in Σ', bounded by $\sigma'(C)$. Deleting a small open disc inside D' that does not meet $\sigma'(G)$, we obtain a cylinder $R' \subseteq S'$ that contains $\sigma'(H')$.

We shall now combine the embeddings $\sigma : G \setminus e \hookrightarrow \Sigma$ and $\sigma' : G \hookrightarrow \Sigma'$ to an embedding $\sigma'' : G \hookrightarrow \Sigma$, which will give us the desired contradiction.

Let $\phi : \sigma'(C) \to \sigma(C)$ be a homeomorphism between the images of C in Σ' and in Σ that commutes with these embeddings, i.e., is such that $\sigma|_C = (\phi \circ \sigma')|_C$. Then extend this to a

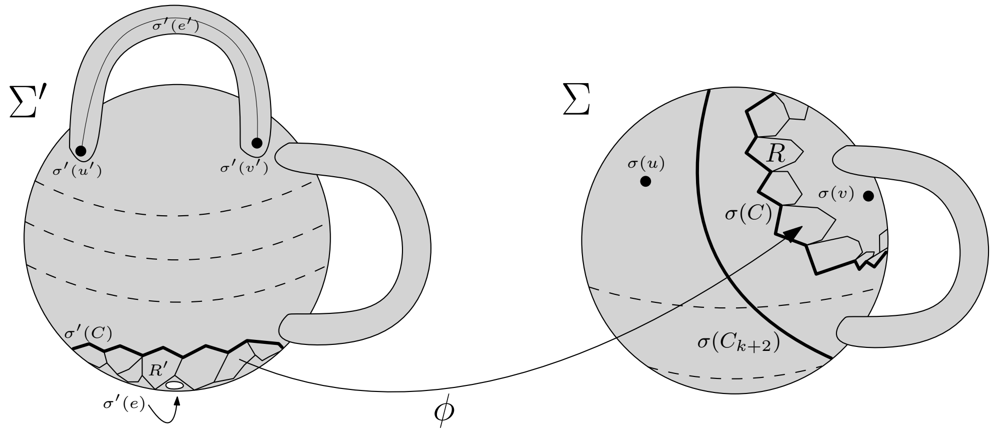

Figure 3.7.4: In the construction of the embedding σ'', we will make use of the homeomorphism ϕ between the cylinders R' and R to embed in the area of Σ that is embedded via σ the cylinder R the subgraph of G which is embedded via σ' in Σ' into the disk D'. This subgraph of G contains also the edge e in which σ is undefined. The rest of G will be embedded by σ'' in Σ in the way that is embedded from σ in Σ but maybe in different parts of Σ.

homeomorphism $\phi : R' \to R$. The idea now is to define σ'' as $\phi \circ \sigma'$ on the part of G which σ' maps to D' (which include the edge e on which σ is undefined), and as σ on the rest of G.

We start by defining the function σ'' on C as $\sigma|_C = (\phi \circ \sigma')|_C$. Next, we define σ'' separately on the components of $G \setminus C$. Since $\sigma'(C)$ bounds the disc D' in Σ', we know that σ' maps each component J of $G \setminus C$ either entirely to D' or entirely to $\Sigma' \setminus D'$. On all the components J such that $\sigma'(J) \subseteq D'$, and on all the edges they send to G, we define σ'' as $\phi \circ \sigma$. Thus, σ'' embeds these components in R. Since $e \in f(H_k) = H' \setminus C$, this includes the component of $G \setminus C$ that contains the edge e.

It remains to define σ'' in the components of $G \setminus C$ which σ' maps to $\Sigma' \setminus D'$. As $\sigma'(C_k) \subseteq D'$, these do not meet C_k. Since $\sigma(C \cup C_k)$ is the frontier of R in S, this means that $\sigma(J) \subseteq \Sigma \setminus R$ or $\sigma(J) \subseteq R$ for every such a component J.

For the component J_0 of $G \setminus C$ that contains C_{k+2} we cannot have $\sigma(J_0) \subseteq R$: as $S_{k+2} \cap R_k = \emptyset$, this would mean that $\sigma(C_{k+2})$ lies in a disc $D \subseteq R$ corresponding to a face of R_k which is impossible since S_{k+2} sends edges to vertices of S_{k+1} outside the boundary of that face. We thus have $\sigma(J_0) \subseteq S \setminus R$, and define σ'' as σ on J_0 and on all $J_0 - C$ edges of G.

Next, consider any remaining component J of $G \setminus C$ that sends no edges to C. If $\sigma(J) \subseteq \Sigma \setminus R$, we define σ'' on J as σ. If $\sigma(J) \subseteq R$ then J is planar. Since J sends no edge to C, we can have σ'' to map J to any open disc in R that has not so far been used by σ''.

It remains to define σ'' in the components $J \neq J_0$ of $G \setminus C$ which σ' maps to $\Sigma' \setminus D'$ and for which G contains a $J - C$ edge. Let $\mathcal{J}$ be the set of all those components J. We shall group them by the way they attach to C, and define σ'' for these groups in turn.

Since $m \geq k+2$, the disc D' lies inside a larger disc in Σ', which is the union of D' and closed discs D'' bounded by the images under $\sigma' \circ f$ of the hexagons in R_{k+1}. By the definition of $\mathcal{J}$, the embedding σ' maps every $J \in \mathcal{J}$ to such a disc D''.

On the path P in C such that $\sigma'(P) = \sigma'(C) \cap D''$ (which is the image under f of one or two consecutive edges on S_{k+1}), let $v_1, \dots v_n$ be the vertices with a neighbor in J_0, in their natural order along P, and write P_i for the segment of P from v_i to v_{i+1}.

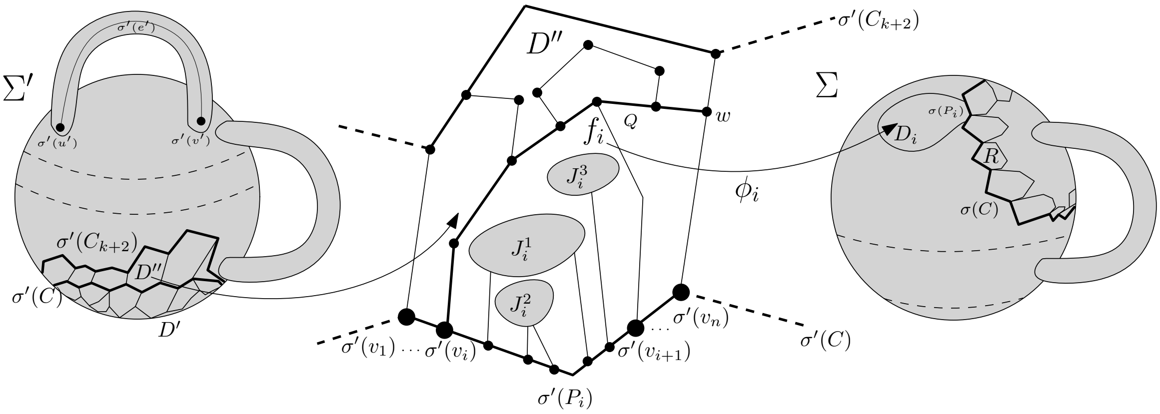

Figure 3.7.5: Illustration of the definition of σ'' on the components $J \neq J_0$ of $G \setminus C$ which σ' maps to $\Sigma' \setminus D'$ and for which G contains an edge witch joins J and C.

Claim 3.7.7. *Let D'' be an arbitrary but fixed closed disc which is bounded by the image under $\sigma' \circ f$ of one hexagon of R_{k+1}. Then, there exists an integer $i \in \{1, \dots, n-1\}$ such that: Every $J \in \mathcal{J}$ with $\sigma'(J) \subseteq D''$ has all its neighbors on C in P_i, and σ' maps J to the face f_i of the plane graph $\sigma'(G[J_0 \cup C]) \cap D''$ whose boundary contains P_i.*

Proof of claim. For any v_i with $1 < i < n$, pick a $v_i - J_0$ edge and extend it though J_0 to a path Q from v_i to C_{k+2} (which exists by the definition of J_0); let w be its first vertex that σ' maps to the boundary circle of D''. By Lemma 1.4.26 applied to $\sigma'(v_i Q w)$ and the two arcs joining $\sigma'(v_i)$ to $\sigma'(w)$ along the boundary circle of D'', there is no arc through D'' that links $\sigma'(P_{i-1})$ to $\sigma'(P_i)$ but avoids $\sigma'(v_i Q w)$. □

We shall define σ'' jointly on all those $J \in \mathcal{J}$ which σ' maps to this f_i, for $i = 1, \dots, n-1$ in turn. To do so, we choose an open disc D_i in $\Sigma \setminus R$ that has a boundary circle containing $\sigma(P_i)$ and avoids the image of σ'' as defined until now. Such D_i exists in a strip neighborhood of $\sigma(C)$ in Σ, because components $J' \in \mathcal{J}$ attaching to a segment $P_j \neq P_i$ of C send no edge to $\mathring{P}_i$. Choose a homeomorphism ϕ_i from the boundary circle of f_i to that of D_i so that $\sigma|_{P_i} = (\phi_i \circ \sigma')|_{P_i}$, and extend this to a homeomorphism ϕ_i from the closure of f_i in S' to the closure of D_i in S. For every $J \in \mathcal{J}$ with $\sigma'(J) \subseteq f_i$, and all $J - C$ edges of G, define σ'' as $\phi_i \circ \sigma'$. □

3.7.2 Characterizing embeddability in any fixed surface

Proof of Theorem 3.1.5. Let Σ be an arbitrary but fixed surface and let *Forb*(Σ) be the set of all the graphs which are not embeddable in Σ and are minimal with respect to the minor relation with this property. Lemma 3.7.3 implies the existence of an positive integer k, which is such that no graph in *Forb*(Σ) has an $(k \times k)$-minor, now by Theorem 3.6.4 there exist a positive integer $f(k)$ such that for each $G \in$ *Forb*(Σ) $\mathbf{bw}(G) \leq k$. Hence the set *Forb*(Σ) is a set of graphs of bounded branch-width, which by Theorem 3.4.2 implies that it is well-quasi-ordered by the minor relation. Since $Forb(\Sigma)$ is an antichain, Theorem 2.1.7, implies its finiteness. □

CHAPTER 4

OTHER GRAPHS' RELATIONS AND WELL-QUASI-ORDERING: A SURVEY OF RESULTS

In this last chapter we survey results regarding the well-quasi-ordering of certain graphs' classes by other than the minor graphs' relations. All -but two- of the examined relations are not well-quasi-ordering graphs in general. Those two exceptions are the weak and the strong immersion, the former has been proved by Neil Robertson and Paul Seymour to be a well-quasi-order for the class of all graphs and the latter has been conjectured from Nash-Williams to be also, but although Robertson and Seymour stated that: *«It seemed to us at one time that we had a proof of the stronger, but even if it was correct it was very much more complicated, and it is unlikely that we will write it down»*, it is still open whether or not graphs are well-quasi-ordered by the strong immersion relation.

4.1 The topological minor relation

As we have already mentioned in the introduction of Chapter 3, the topological minor relation does not well-quasi-ordering graphs in general and is the first graphs' relation where studied extensively from the perspective of well-quasi-ordering. Recall the conjecture that Vázsonyi made in 1937[1] for the well-quasi-ordering of finite trees which was independently proved in 1960 by Kruskal [81] and Tarkowski [116] with a shorter proof given by Nash-Williams [95] in 1963. Nash-Williams [97] also generalized the Tree Theorem by proving the well-quasi-ordering of all trees -finite and infinite- by the topological minor relation in 1965, which was a Kruskal's conjecture ([81, Conjecture 1]).

Theorem 4.1.1 (Nash-Williams [97]). The set of all trees (finite or infinite) is well-quasi-ordered by the topological minor relation.

In addition to these results Mader [90] proved in 1972 Theorem 3.1.17 -by making use of the structural characterization of graphs without k disjoint cycles which was given by Erdős and Pósa in [43] in 1965- which states that for every fixed positive integer k the set of all graphs with no

[1] The year of the conjecture is remarked by Lovász [88]

k disjoint cycles is well-quasi-ordered by the topological minor relation. Vázsonyi made also the following conjecture:

Conjecture 4.1.2 (Vázsonyi's conjecture)**.** The set of all subcubic graphs is well-quasi-ordered by the topological minor relation.

Recall that the topological minor relation is the same with the minor relation on subcubic graphs (Proposition 1.2.59). Hence Robertson and Seymour's theorem implies the above Vázsonyi's conjecture.

For each positive natural number n let A_n be the graph illustrated in Figure 4.1.1. In 1996 Ding [36] proved the following.

Theorem 4.1.3 (Ding [36])**.** Let $\mathcal{G}$ be a minor-closed class of graphs. Then $\mathcal{G}$ is well-quasi-ordered by the topological minor relation if and only if the intersection of $\mathcal{G}$ and $\mathcal{A}$ is finite, where $\mathcal{A} = \{A_n | n \in \mathbb{N}\}$.

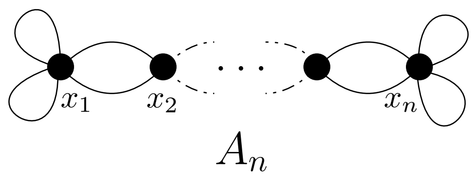

Figure 4.1.1: The graph A_n of the statement of Ding's Theorem 4.1.3.

Definition 4.1.4 (feedback vertex set)**.** Let G be a graph and $X \subseteq V(G)$. The set X will be said to be a *feedback vertex set* of G, if and only if the graph $G[V(G) \setminus X]$ contains no cycles.

After the above definition we can restate the Erdős-Pósa Theorem 4.1.5 as follows:

Theorem 4.1.5. Given a natural number k there exists a natural number k' such that for every graph G

- Either G has k vertex-disjoint cycles, or
- G has a feedback vertex set of order at most k'.

In 2012 Fellows, Hermelin, and Rosamond [48] gave a proof of the following theorem which -due to the above restatement of the Erdős-Pósa Theorem 4.1.5- is equivalent with Mader's Theorem 3.1.17 but their proof is considerably simpler than the original proof of Mader.

Theorem 4.1.6 ([48, Theorem 1])**.** Let k be a natural number, then the set of all graphs which have a feedback vertex set of order at most k is well-quasi-ordered by the topological minor relation.

Definition 4.1.7 (Robertson chain of length k, ends of the Robertson chain of length k)**.** Let k be a positive natural number, consider the graph which we obtain by doubling the edges of the path of length k, any graph isomorphic to this graph will be said to be a *Robertson chain of length* k. The vertices which correspond to the ends of the original path will be said to be the *ends of the Robertson chain*.

In the late 1980’s, Robertson conjectured that the Robertson chain is the only obstruction for the well-quasi-ordering of graphs by the topological minor relation.

Conjecture 4.1.8 (Robertson’s conjecture)**.** For every positive integer k, the set of all graphs with no topological minor isomorphic to the path of length k with each edge duplicated, is well-quasi-ordered by the topological minor relation.

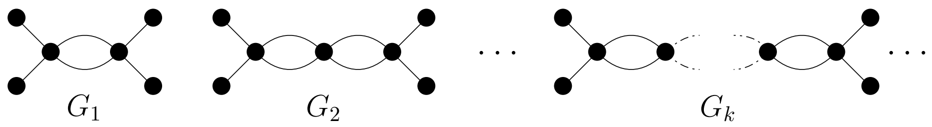

Figure 4.1.2: An infinite antichain of graphs with respect to the topological minor relation.

In 2014 in his PhD thesis Liu [87], proved Robertson Conjecture 4.1.8.

Theorem 4.1.9 (Liu [87])**.** For every positive integer k, the set of all graphs with no topological minor isomorphic to the path of length k with each edge duplicated, is well-quasi-ordered by the topological minor relation.

Actually Liu proved the following stronger theorem:

Theorem 4.1.10 (Liu [87])**.** Let k, l, be nonnegative integers, the set of all graphs that contain at most l different topological minors isomorphic to the Robertson chain of length k is well-quasi-ordered by the topological minor relation.

Liu's Theorem 4.1.10 has the following immediate corollary:

Corollary 4.1.11. Let $\mathcal{Q}$ be a graph property which is closed under the topological minor relation, then for every positive integer k there exists a positive integer n and graphs $H_1, \dots, H_n$, such that for every graph G that does not contain a topological minor isomorphic to the Robertson chain of length k, G satisfies the property $\mathcal{Q}$ ($G \in \mathcal{Q}$) if and only if for every $i \in \{1, \dots, n\}$ the graph G has no topological minor isomorphic to the graph H_i.

Grohe, Kawarabayashi, Marx, and Wollan [62] proved in 2011 that topological minor containment is polynomial-time decidable.

Theorem 4.1.12 (Grohe, Kawarabayashi, Marx, and Wollan [62, Theorem 1.1])**.** For every fixed graph H, there is a $\mathcal{O}(|V(H)|^3)$-time algorithm that decides if H is a topological minor of G.

The above theorem together with Corollary 4.1.11 have the following immediate corollary:

Corollary 4.1.13. For every fixed positive integer k, testing any topological minor-closed property in the class of graphs that do not contain a topological minor isomorphic to the Robertson chain of length k can be done in polynomial time.

Although we do not state it here, we remark that Grohe and Marx [61] proved in 2011 a structure theorem which describes the "rough" structure of graphs which exclude a fixed graph as a topological minor and thus they generalized the structure theorem of Robertson and Seymour [113]. A shorter proof of this theorem was given by Erde and Weißauer [41] in 2019.

4.2 The weak and the strong immersion relations

In 1963 Nash-Williams [96] conjectured that graphs are well-quasi-ordered by the weak immersion relation and in 1965 [97] he made the analogue conjecture for the strong immersion relation, which is still widely open.

Conjecture 4.2.1 (Nash-Williams [97]). The class of all graphs is well-quasi-ordered by the strong immersion relation.

Conjecture 4.2.2 (Nash-Williams [96]). The class of all graphs is well-quasi-ordered by the weak immersion relation.

The conjecture considering the weak immersion relation was proved by Robertson and Seymour [103] in the last paper of their Graph Minors series [104] in 2010.

Theorem 4.2.3 (Robertson and Seymour [103, Theorem 1.1]). The class of all graphs is well-quasi-ordered by the weak immersion relation.

Robertson and Seymour's above theorem have the following immediate corollary:

Corollary 4.2.4. Let $\mathcal{Q}$ be a graph property which is closed under the weak immersion relation, then there exists a positive integer n and graphs $H_1, \ldots, H_n$, such that an arbitrary graph G satisfies the property $\mathcal{Q}$ ($G \in \mathcal{Q}$) if and only if $(\forall i \in \{1, \ldots, n\})[H_i \not\leq_{wi} G]$.

One year after the proof of Theorem 4.2.3, Grohe, Kawarabayashi, Marx, and Wollan [62] obtained the following result:

Theorem 4.2.5 (Grohe, Kawarabayashi, Marx, and Wollan [62, Corollary 1.2]). For every fixed graph H, there is a $\mathcal{O}(|V(H)|^3)$-time algorithm that decides if there is an immersion of H into G.

The above theorem together with Corollary 4.2.4 has the following immediate corollary.

Corollary 4.2.6. Every weak immersion-closed property of graphs can be tested in polynomial time.

For the strong immersion relation Andreae [1] proved in 1986 the following theorem:

Theorem 4.2.7 (Andreae [1]). The following classes of graphs are well-quasi-ordered by the strong immersion relation:

(i) The class of all simple graphs that do not contain $K_{2,3}$ as a strong immersion.

(ii) The class of all graphs whose blocks are complete graphs, cycles or complete bipartite graphs.

Below we survey some results considering the weak and the strong immersion relations on directed graphs.

Definition 4.2.8 (weak and strong immersion relations on directed graphs). Let G, H be directed graphs. A *weak immersion* of H in G is a map η such that the following conditions hold:

(i) $(\forall v \in V(H))[\eta(v) \in V(G)]$;

(ii) $(\forall v, u \in V(H))[v \neq u \Rightarrow \eta(v) \neq \eta(u)]$;

(iii) for each edge $e = (u, v)$ of H, $\eta(e)$ is a directed path of G from $\eta(u)$ to $\eta(v)$;

(iv) and if $e, f \in E(H)$ are distinct, then the paths of G $\eta(e)$ and $\eta(f)$ have no edges in common although they may share vertices.

If in addition we add the condition:

(v) if $v \in V(H)$ and $e \in E(H)$, and e is not incident with v in H, then $\eta(v)$ is not a vertex of the path $\eta(e)$.

we call the relation *strong immersion*.

Observation 4.2.9. The class of all directed graphs is not well-quasi-ordered by the weak immersion relation. In order to see that observe that the set $\{C_{2k} | (k \in \mathbb{N}) \wedge (k \geq 2)\}$ where C_{2k} is a cycle on $2k$ vertices with its edges alternately oriented clockwise and counter-clockwise, is an infinite antichain of directed graphs with respect to the weak immersion relation.

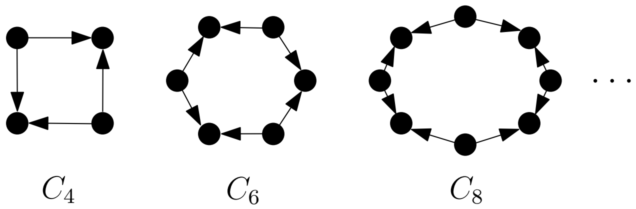

Figure 4.2.1: An infinite antichain of directed graphs for both the weak and the strong immersion relations.

Definition 4.2.10 (tournament)**.** A directed graph G is said to be a *tournament* if and only if it is a complete graph, i.e. $(\forall v, u \in V(G))[((u, v) \in E(G)) \wedge ((v, u) \in E(G))]$.

Chudnovsky and Seymour [20] in 2011 proved the following:

Theorem 4.2.11 (Chudnovsky and Seymour [20])**.** The set of all tournaments is well-quasi-ordered by the strong immersion relation.

Definition 4.2.12 (semi-complete directed graph)**.** A directed graph G is said to be *semi-complete* if and only if $(\forall v, u \in V(G))[((v, u) \in E(G)) \vee ((u, v) \in E(G))]$.

Since the class of semi-complete directed graphs contain tournaments, Barbero, Paul, and Pilipczuk [9] in 2018 generalized the aforementioned result of Chudnovsky and Seymour [20] by proving the following:

Theorem 4.2.13 (Barbero, Paul, and Pilipczuk [9])**.** The set of all semi-complete directed graphs is well-quasi-ordered by the strong immersion relation.

Theorem 4.2.14 (Chudnovsky, Fradkin, and Seymour [22])**.** For every directed graph H there is an algorithm which for every semi-complete directed graph G decides in $\mathcal{O}(|V(G)|^3)$ whether there is a strong or weak immersion of H in G

Corollary 4.2.15. For every directed graph property $\mathcal{Q}$ which is closed under the strong immersion relation there exists a cubic time algorithm which presented a semi-complete directed graph D, decides whether or not $D \in \mathcal{Q}$.

4.3 The subgraph and the induced subgraph relations

The set of all graphs is not well-quasi-ordered neither by the subgraph relation nor by the induced subgraph relation, in Figure 4.3.1 is illustrated an infinite antichain of graphs for both the subgraph and the induce subgraph relations, but there are several positive results -a lot of them we survey below- if we restrict the problem to smaller classes of graphs.

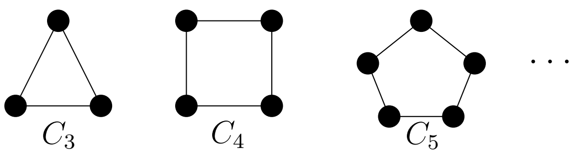

Figure 4.3.1: The set of all cycles consist an infinite antichain of graphs for both the subgraph and the induce subgraph relations.

Theorem 4.3.1 (Damaschke [27])**.** The set of all graphs that do not contain a graph isomorphic to P_4 as an induced subgraph is well-quasi-order by the induced subgraph relation.

Theorem 4.3.2 (Damaschke [27])**.** The set of all graphs that do not contain neither a graph isomorphic to K_3 nor a graph isomorphic to the disjoint union of K_2 and two copies of K_1 as an induced subgraph is well-quasi-order by the induced subgraph relation.

Theorem 4.3.3 (Damaschke [27])**.** The set of all graphs that do not contain neither a graph isomorphic to K_3 nor a graph isomorphic to P_5 as an induced subgraph is well-quasi-order by the induced subgraph relation.

Theorem 4.3.4 (Ding [35])**.** Let k be a positive integer. The set of all graphs that do not contain a subgraph isomorphic to P_k is well-quasi-ordered by the induced subgraph relation.

Theorem 4.3.5 (Ding [35])**.** Let k be a positive integer. The set of all directed graphs that their underlying graph do not contain a subgraph isomorphic to P_k is well-quasi-ordered by the induced subgraph relation.

Ding [35] proved also the following negative result.

Theorem 4.3.6 (Ding [35])**.** The set of all graphs that do not contain a graph isomorphic to P_5 as an induced subgraph is not well-quasi-ordered by the induced subgraph relation.

Ding [35] proved that essentially the cycles $C_3, C_4, \ldots$ and the graphs $H_1, H_2, H_3, \ldots$ are the only two infinite antichains with respect to the subgraph relation. More formally, Ding proved that a class of graphs closed under taking subgraphs is well-quasi-ordered by the subgraph relation if and only if it contains finitely many graphs C_n and H_n.

Definition 4.3.7 (ideal with respect a quasi-order)**.** Let X be a set that is quasi-ordered by a relation $\leq$ and $I \subseteq X$. The set I will be said to be an *ideal* (w.r.t. $\leq$) if and only if $(\forall x, y \in X)[(x \leq y) \wedge (y \in I) \Rightarrow x \in I]$.

Theorem 4.3.8 (Ding [35])**.** Let I be an ideal of graphs with respect to the subgraph relation. Then the following are equivalent:

(i) The set I is well-quasi-ordered by the subgraph relation.

(ii) The set I is well-quasi-ordered by the induced subgraph relation.

(iii) There exists a positive integer k such that I does not contain any cycle of length greater or equal to k and any graph that can be obtained from a path of length greater or equal to k by attaching two vertices to each end of the path.

Definition 4.3.9 (vertex cover of a graph)**.** Let G be a graph a $X \subseteq V(G)$ will be said to be a *vertex cover* of G if and only if $(\forall e \in E(G))[e \cap X \neq \emptyset]$.

Theorem 4.3.10 (Fellows, Hermelin, and Rosamond [48])**.** Let k be a positive integer. The set of all graphs that have a vertex cover of size at most k is well-quasi-ordered by the induced subgraph relation.

Below and at the next sections we state some results which involve terms which are not defined in the present.

Theorem 4.3.11 (Korpelainen, Atminas, Brignall, Vatter, and Lozin [79])**.** Let k be a positive integer. The set of all permutation graphs that do not contain neither a graph isomorphic to P_5 nor a graph isomorphic to K_k as an induced subgraph is well-quasi-order by the induced subgraph relation.

Korpelainen, Atminas, Brignall, Vatter, and Lozin [79] proved the following negative result:

Theorem 4.3.12 (Korpelainen, Atminas, Brignall, Vatter, and Lozin [79])**.** The following three sets of graphs are not well-quasi-ordered by the induced subgraph relation:

(i) The set of all permutation graphs that do not contain neither a graph isomorphic to P_6 nor a graph isomorphic to K_6 as an induced subgraph.

(ii) The set of all permutation graphs that do not contain neither a graph isomorphic to P_7 nor a graph isomorphic to K_5 as an induced subgraph.

(iii) The set of all permutation graphs that do not contain neither a graph isomorphic to P_8 nor a graph isomorphic to K_4 as an induced subgraph.

Theorem 4.3.13 (Petkovšek [99])**.** Let k be a positive integer. The set of all k-letter graphs is well-quasi-ordered by the induced subgraph relation.

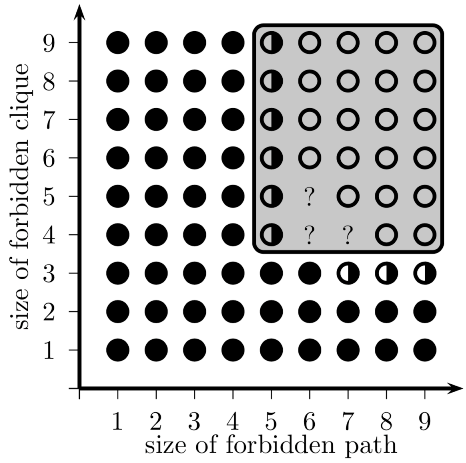

Figure 4.3.2: The figure illustrated above is from [79] and shows the known results regarding well-quasi-ordering from $\leq_{is}$ for sets of graphs and sets of permutation graphs avoiding paths and cliques, including the results of [79]. Filled circles indicate that all graphs avoiding the specified path and clique are well-quasi-ordered by $\leq_{is}$. Half-filled circles indicate that the corresponding class of permutation graphs are well-quasi-ordered by $\leq_{is}$, but that the corresponding class of all graphs are not. Empty circles indicate that neither class is well-quasi-ordered by $\leq_{is}$. For the three unknown cases (indicated by question marks), it is known that the corresponding class of graphs contains an infinite antichain w.r.t. $\leq_{is}$.

Theorem 4.3.14 (Korpelainen and Lozin [78]). The set of all bipartite graphs that do not contain a graph isomorphic to P_7 as an induced subgraph is not well-quasi-ordered by the induced subgraph relation.

Theorem 4.3.15 (Korpelainen and Lozin [78]). The following sets of graphs are well-quasi-ordered by the induced subgraph relation:

(i) The set of all bipartite graphs that do not contain neither a graph isomorphic to P_7 nor a graph isomorphic to

(ii) The set of all bipartite graphs that do not contain neither a graph isomorphic to P_7 nor a graph isomorphic to C_4 as an induced subgraph.

(iii) For any positive integer k, the set of all bipartite permutation graphs that do not contain a graph isomorphic to P_k as an induced subgraph.

Other results considering the well-quasi-ordering by the subgraph and the induced subgraph relations can be found at [5, 6, 7, 21, 26, 48, 77].

4.4 The induced minor relation

The set of all graphs is not well-quasi-ordered by the induced minor relation, Figure 4.4.1 illustrates an infinite antichain of graphs with respect to the induced minor relation.

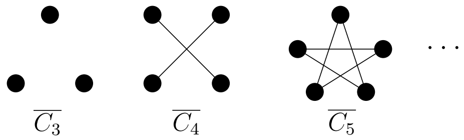

Figure 4.4.1: An infinite antichain of graphs with respect to the induced minor relation.

Thomas [118] in 1985 proved that planar graphs are not well-quasi-ordered by the induced minor relation but series-parallel graphs are.

Theorem 4.4.1 (Thomas [118])**.** The set of all series-parallel graphs is well-quasi-ordered by the induced minor relation.

Theorem 4.4.2 (Thomas [118])**.** The set of all graphs with no induced minor isomorphic to K_4 is well-quasi-ordered by the induced minor relation.

Definition 4.4.3 (clique number of a graph)**.** Let G be a graph, the *clique number* of G is the maximum positive integer n such that G has a K_n-subgraph.

Ding [37] proved in 1998 the following theorems.

Theorem 4.4.4 (Ding [37])**.** For every positive integer n the set of all chordal graphs with clique number at most n is well-quasi-ordered by the induced minor relation

Theorem 4.4.5 (Ding [37])**.** The set of all interval graphs is not well-quasi-ordered by the induced minor relation.

The following is due to Fellows, Hermelin, and Rosamond [48].

Theorem 4.4.6 (Fellows, Hermelin, and Rosamond [48])**.** Let k be a positive integer. The set of all graphs with circumference at most k is well-quasi-ordered by the induced minor relation.

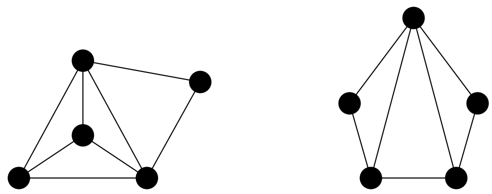

Figure 4.4.2: The graph K_4^+ on the left hand side and the graph Gem in the right hand side.

Błasiok, Kamiński, Raymond, and Trunck [15] in 2015 gave a complete characterization of graphs H such that the set of H-induced-minor-free graphs is well-quasi-ordered by the induced minor relation.

Theorem 4.4.7 (Błasiok, Kamiński, Raymond, and Trunck [15])**.** Given a graph H the set of all graphs with no induced minor isomorphic to H is well-quasi-ordered by the induced minor relation if and only if H is an induced minor of K_4^+ or of the Gem.

4.5 The contraction relation

The set of all graphs is not well-quasi-ordered by the induced minor relation, in Figure 4.5.1 it is illustrated an infinite antichain of graphs with respect to the contraction relation.

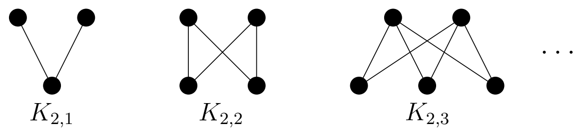

Figure 4.5.1: An infinite antichain of graphs with respect to the contraction relation.

Kamiński, Raymond, and Trunck [66] in 2016 gave a complete characterization of graphs H such that the set of H-contraction-free graphs is well-quasi-ordered by the contraction relation.

Definition 4.5.1 (diamond graph)**.** A graph will be said to be a *diamond* graph if and only if it is isomorphic to the graph obtained from K_4 if we delete an arbitrary edge.

Theorem 4.5.2 (Kamiński, Raymond, and Trunck [66])**.** The class of connected H-contraction-free graphs is well-quasi-ordered by contractions if and only if H is a contraction of the diamond.

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